# FabULous Interoperability for ML and a Linear Language

## Supplementary Material

ANONYMOUS FOR SUBMISSION

**ACM Reference format:**

Anonymous for submission. 2016. FabULous Interoperability for ML and a Linear Language. 1, 1, Article 1 (January 2016), 25 pages.
DOI: 10.1145/nnnnnnn.nnnnnnn

## 1 UL LOGICAL RELATION

If $\mathrm{dom}(\gamma_1) = \Gamma_1, \mathrm{dom}(\gamma_2) = \Gamma_2$, and $\Gamma = \Gamma_1 \boxplus \Gamma_2$, then $\gamma_1 \boxplus \gamma_2$ is defined when for any variable $x \in \Gamma_1 \cap \Gamma_2$, $\gamma_1(x) = \gamma_2(x)$ and is defined as $\gamma(x) = \gamma_1(x)$ if $x \in \Gamma_1$ and $\gamma(x) = \gamma_2(x)$ if $x \in \Gamma_2$.

$(\mathsf{R}, \sigma_1, \sigma_2)$ is well formed if $\sigma_1, \sigma_2$ are closed types and $\mathsf{R} \in \mathrm{Rel}[\sigma_1, \sigma_2]$. The relations $\mathcal{V}[\![\rho]\!]^j, \mathcal{E}[\![\rho]\!]^j, \ldots$ below are only defined for *closed* relation types $\rho, \rho$. Substitution is extended to $\rho, \rho$ by considering $(\mathsf{R}, \sigma_1, \sigma_2)$ to be closed.

Every $\rho$ has two associated types, the types of terms that it relates, which we denote $(\rho)_1, (\rho)_2$. It is defined as follows:

| | | | | | |
|---|---|---|---|---|---|
| $((\mathsf{R}, \sigma_1, \sigma_2))_1$ | $\stackrel{\mathrm{def}}{=}$ | $\sigma_1$ | $(\rho_1 \otimes \rho_2)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\rho_1 \otimes \rho_2$ |
| $((\mathsf{R}, \sigma_1, \sigma_2))_2$ | $\stackrel{\mathrm{def}}{=}$ | $\sigma_2$ | $(1)_i$ | $\stackrel{\mathrm{def}}{=}$ | $1$ |
| $(\alpha)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\alpha$ | $(\rho_1 \multimap \rho_2)_i$ | $\stackrel{\mathrm{def}}{=}$ | $(\rho_1)_i \multimap (\rho_2)_i$ |
| $(\rho_1 \times \rho_2)_i$ | $\stackrel{\mathrm{def}}{=}$ | $(\rho_1)_i \times (\rho_2)_i$ | $(\rho_1 \oplus \rho_2)_i$ | $\stackrel{\mathrm{def}}{=}$ | $(\rho_1)_i \oplus (\rho_2)_i$ |
| $(1)_i$ | $\stackrel{\mathrm{def}}{=}$ | $1$ | $(\mu\alpha.\, \rho)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\mu\alpha.\, (\rho)_i$ |
| $(\rho_1 \to \rho_2)_i$ | $\stackrel{\mathrm{def}}{=}$ | $(\rho_1)_i \to (\rho_2)_i$ | $(\alpha)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\alpha$ |
| $(\rho_1 + \rho_2)_i$ | $\stackrel{\mathrm{def}}{=}$ | $(\rho_1)_i + (\rho_2)_i$ | $(!\rho)_i$ | $\stackrel{\mathrm{def}}{=}$ | $!(\rho)_i$ |
| $(\mu\alpha.\, \rho)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\mu\alpha.\, (\rho)_i$ | $(\mathsf{Box}\ 1\ \rho)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\mathsf{Box}\ 1\ (\rho)_i$ |
| $(\forall\alpha.\, \rho)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\forall\alpha.\, (\rho)_i$ | $(\mathsf{Box}\ 0)_i$ | $\stackrel{\mathrm{def}}{=}$ | $\mathsf{Box}\ 0$ |

2017. Manuscript submitted to ACM

$$
\begin{aligned}
\text{Atom}[\sigma] &\overset{\text{def}}{=} \{\mathsf{v} \mid \cdot \vdash_{\mathsf{U}} \mathsf{v} : \sigma\} \\
\text{Rel}[\sigma_1, \sigma_2] &\overset{\text{def}}{=} \{\mathsf{R} : \mathbb{N} \to \mathcal{P}(\text{Atom}[\sigma_1] \times \text{Atom}[\sigma_2]) \mid \forall j \leq j'.\ \mathsf{R}^{j'} \subset \mathsf{R}^{j}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}\,[\![(\mathsf{R}, \sigma_1, \sigma_2)]\!]^{j} &\overset{\text{def}}{=} \mathsf{R}^{j} \\
\mathcal{V}\,[\![1]\!]^{j} &\overset{\text{def}}{=} \{(\langle\rangle, \langle\rangle)\} \\
\mathcal{V}\,[\![\rho \times \rho']\!]^{j} &\overset{\text{def}}{=} \{(\langle \mathsf{v}_1, \mathsf{v}'_1\rangle, \langle \mathsf{v}_2, \mathsf{v}'_2\rangle) \mid (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho]\!]^{j} \wedge (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho]\!]^{j}\} \\
\mathcal{V}\,[\![\rho_1 + \rho_2]\!]^{j} &\overset{\text{def}}{=} \{(\mathsf{inj_i}\, \mathsf{v}_1, \mathsf{inj_i}\, \mathsf{v}_2) \mid (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho_i]\!]^{j}\} \\
\mathcal{V}\,[\![\mu\alpha.\,\rho]\!]^{j} &\overset{\text{def}}{=} \{(\mathsf{fold}_{(\mu\alpha.\rho)_1}\, \mathsf{v}_1, \mathsf{fold}_{(\mu\alpha.\rho)_2}\, \mathsf{v}_1) \mid \forall j' < j.\ (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho[\mu\alpha.\,\rho/\alpha]]\!]^{j'}\} \\
\mathcal{V}\,[\![\rho_1 \to \rho_2]\!]^{j} &\overset{\text{def}}{=} \{(\lambda(\mathsf{x}_1 : (\rho_1)_1).\, \mathsf{e}_1, \lambda(\mathsf{x}_2 : (\rho_1)_2).\, \mathsf{e}_2) \mid \\
&\qquad \forall j' \leq j, (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho_1]\!]^{j'}.\ (\mathsf{e}_1[\mathsf{v}_1/\mathsf{x}_1], \mathsf{e}_2[\mathsf{v}_2/\mathsf{x}_2]) \in \mathcal{E}\,[\![\rho_2]\!]^{j'}\} \\
\mathcal{V}\,[\![\forall\alpha.\,\rho]\!]^{j} &\overset{\text{def}}{=} \{(\Lambda\alpha.\,\mathsf{v}_1, \Lambda\alpha.\,\mathsf{v}_2) \mid \forall \sigma_1, \sigma_2, \mathsf{R} \in \text{Rel}[\sigma_1, \sigma_2].\ (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho[(\mathsf{R}, \sigma_1, \sigma_2)/\alpha]]\!]^{j}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}\,[\![\rho]\!]^{j} &\overset{\text{def}}{=} \{(\mathsf{e}_1, \mathsf{e}_2) \mid \forall j' \leq j.\ \mathsf{e}_1 \overset{\mathsf{U}}{\hookrightarrow}{}^{j'} \mathsf{v}_1 \Rightarrow \\
&\qquad \exists \mathsf{v}_2.\ \mathsf{e}_2 \overset{\mathsf{U}}{\hookrightarrow}{}^{*} \mathsf{v}_2 \wedge (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\rho]\!]^{j-j'}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}[\![1]\!]^j &\stackrel{\text{def}}{=} \{((\emptyset \mid \langle\rangle), (\emptyset \mid \langle\rangle))\} \\
\mathcal{V}[\![\rho \otimes \rho']\!]^j &\stackrel{\text{def}}{=} \{((s_1 + s_1' \mid \langle v_1, v_1'\rangle), (s_2 + s_2' \mid \langle v_2, v_2'\rangle)) \mid \\
&\qquad ((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\rho]\!]^j \wedge ((s_1' \mid v_1'), (s_2' \mid v_2')) \in \mathcal{V}[\![\rho']\!]^j\} \\
\mathcal{V}[\![\rho_1 \oplus \rho_2]\!]^j &\stackrel{\text{def}}{=} \{((s_1 \mid \mathsf{inj}_i\, v_1), (s_2 \mid \mathsf{inj}_i\, v_2)) \mid \\
&\qquad ((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\rho_i]\!]^j\} \\
\mathcal{V}[\![\mu\alpha.\,\rho]\!]^j &\stackrel{\text{def}}{=} \{((s_1 \mid \mathsf{fold}_{\mu\alpha.\rho}\, v_1), (s_2 \mid \mathsf{fold}_{\mu\alpha.\rho}\, v_2)) \mid \\
&\qquad \forall j' < j.((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\rho[\mu\alpha.\,\rho/\alpha]]\!]^{j'}\} \\
\mathcal{V}[\![\rho' \multimap \rho]\!]^j &\stackrel{\text{def}}{=} \{((s_1 \mid \lambda(x:\rho').\, e_1), (s_2 \mid \lambda(x:\rho').\, e_2)) \mid \\
&\qquad \forall j' \leq j, s_1', s_2', ((s_1'' \mid v_1), (s_2'' \mid v_2)) \in \mathcal{V}[\![\rho']\!]^{j'}. \\
&\qquad\quad s_1' = s_1 + s_1'' \wedge s_2' = s_2 + s_2'' \Rightarrow \\
&\qquad\quad ((s_1' \mid e_1[v_1/x]), (s_2' \mid e_2[v_2/x])) \in \mathcal{E}[\![\rho]\!]^{j'}\} \\
\mathcal{V}[\![!\rho]\!]^j &\stackrel{\text{def}}{=} \{((\emptyset \mid \mathsf{share}(s_1 : \Psi_1).\, v_1), (\emptyset \mid \mathsf{share}(s_2 : \Psi_2).\, v_2)) \mid \\
&\qquad ((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\rho]\!]^j\} \\
\mathcal{V}[\![\mathsf{Box}\ 0]\!]^j &\stackrel{\text{def}}{=} \{(([\ell_1 \mapsto \cdot] \mid \ell_1), ([\ell_2 \mapsto \cdot] \mid \ell_2))\} \\
\mathcal{V}[\![\mathsf{Box}\ 1\ \rho]\!]^j &\stackrel{\text{def}}{=} \{(([\ell_1 \mapsto (s_1 \mid v_1)] \mid \ell_1), ([\ell_2 \mapsto (s_2 \mid v_2)] \mid \ell_2)) \mid \\
&\qquad ((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\rho]\!]^j\} \\
\mathcal{V}[\![[\rho]]\!]^j &\stackrel{\text{def}}{=} \{((\emptyset \mid [v_1]), (\emptyset \mid [v_2])) \mid (v_1, v_2) \in \mathcal{V}[\![\rho]\!]^j\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}[\![\rho]\!]^j &\stackrel{\text{def}}{=} \{((s_1 \mid e_1), (s_2 \mid e_2)) \mid \\
&\qquad \forall j' \leq j, (s_1' \mid v_1).(s_1 \mid e_1) \overset{\mathsf{L}}{\hookrightarrow}{}^{j'} (s_1' \mid v_1) \Rightarrow \\
&\qquad\quad \exists (s_2' \mid v_2).(s_2 \mid e_2) \overset{\mathsf{L}}{\hookrightarrow}{}^{*} (s_2' \mid v_2) \wedge \\
&\qquad\quad\quad ((s_1' \mid v_1), (s_2' \mid v_2)) \in \mathcal{V}[\![\rho]\!]^{j-j'}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}[\![\cdot]\!]^j &\stackrel{\text{def}}{=} \{((\emptyset, \emptyset) \mid \varnothing)\} \\
\mathcal{G}[\![\Gamma, x:\sigma]\!]^j &\stackrel{\text{def}}{=} \{((s_1 + s_1', s_2 + s_2') \mid \gamma[x \mapsto (v_1, v_2)]) \mid \\
&\qquad ((s_1, s_2) \mid \gamma) \in \mathcal{G}[\![\Gamma]\!]^j \wedge ((s_1' \mid v_1), (s_2' \mid v_2)) \in \mathcal{V}[\![(\gamma)_R(\sigma)]\!]^j\} \\
\mathcal{G}[\![\Gamma, x:\sigma]\!]^j &\stackrel{\text{def}}{=} \{((s_1, s_2) \mid \gamma[x \mapsto (v_1, v_2)]) \mid \\
&\qquad ((s_1, s_2) \mid \gamma) \in \mathcal{G}[\![\Gamma]\!]^j \wedge (v_1, v_2) \in \mathcal{V}[\![(\gamma)_R(\sigma)]\!]^j\} \\
\mathcal{G}[\![\Gamma, \alpha]\!]^j &\stackrel{\text{def}}{=} \{((s_1, s_2) \mid \gamma[\alpha \mapsto (R, \sigma_1, \sigma_2)]) \mid \\
&\qquad R \in \mathrm{Rel}[\sigma_1, \sigma_2] \wedge ((s_1, s_2) \mid \gamma) \in \mathcal{G}[\![\Gamma]\!]^j\}
\end{aligned}
$$

$$\boxed{\Gamma_1 \boxplus \Gamma_2}$$

$$\begin{array}{lcll}
(\Gamma_1, x{:}!\sigma) \boxplus (\Gamma_2, x{:}!\sigma) & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), x{:}!\sigma & \\
(\Gamma_1, x{:}\sigma) \boxplus \Gamma_2 & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), x{:}\sigma & (x \notin \Gamma_2) \\
\Gamma_1 \boxplus (\Gamma_2, x{:}\sigma) & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), x{:}\sigma & (x \notin \Gamma_1) \\
(\Gamma_1, x{:}\sigma) \boxplus (\Gamma_2, x{:}\sigma) & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), x{:}\sigma & \\
(\Gamma_1, x{:}\sigma) \boxplus \Gamma_2 & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), x{:}\sigma & (x \notin \Gamma_2) \\
\Gamma_1 \boxplus (\Gamma_2, x{:}\sigma) & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), x{:}\sigma & (x \notin \Gamma_1) \\
(\Gamma_1, \alpha) \boxplus (\Gamma_2, \alpha) & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), \alpha & \\
(\Gamma_1, \alpha) \boxplus \Gamma_2 & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), \alpha & (\alpha \notin \Gamma_2) \\
\Gamma_1 \boxplus (\Gamma_2, \alpha) & \stackrel{\text{def}}{=} & (\Gamma_1 \boxplus \Gamma_2), \alpha & (\alpha \notin \Gamma_1)
\end{array}$$

Fig. 1. Multilanguage Context Merging

$$\begin{array}{lcl}
\rho & ::= & (R, \sigma_1, \sigma_2) \mid \alpha \mid \rho_1 \times \rho_2 \mid 1 \mid \rho_1 \to \rho_2 \mid \rho_1 + \rho_2 \mid \mu\alpha.\, \rho \mid \forall \alpha.\, \rho \\
\rho & ::= & \rho_1 \otimes \rho_2 \mid 1 \mid \rho_1 \multimap \rho_2 \mid \rho_1 \oplus \rho_2 \mid \mu\alpha.\, \rho \mid \alpha \mid !\rho \mid \mathsf{Box}\ 1\ \rho \mid \mathsf{Box}\ 0
\end{array}$$

Fig. 2. Relation Type Syntax

$$!\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma \stackrel{\text{def}}{=} \forall j \geq 0, ((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma]\!]^j . ((\gamma)_1(e_1), (\gamma)_2(e_2)) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^j$$

$$!\Gamma \vdash_v v_1 \lesssim^{log} v_2 : \sigma \stackrel{\text{def}}{=} \forall j \geq 0, ((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma]\!]^j . ((\gamma)_1(v_1), (\gamma)_2(v_2)) \in \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^j$$

$$\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma \stackrel{\text{def}}{=} \forall j \geq 0, ((s_1', s_2') \mid \gamma) \in \mathcal{G}\,[\![\Gamma]\!]^j . \\ ((s_1 + s_1' \mid (\gamma)_1(e_1)), (s_2 + s_2' \mid (\gamma)_2(e_2))) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^j$$

Fig. 3. Logical Approximation for Open Terms

$$!\Gamma \vdash e_1 \lesssim^{ctx} e_2 : \sigma \stackrel{\text{def}}{=} \forall C.\, \cdot \vdash_U C[e_1] : 1 \wedge \cdot \vdash_U C[e_2] : 1 \wedge C[e_1] \stackrel{U}{\hookrightarrow}^* \langle\rangle \implies C[e_2] \stackrel{U}{\hookrightarrow}^* \langle\rangle$$

$$\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{ctx} (s_2 \mid e_2) : \sigma \stackrel{\text{def}}{=} \forall C.\, \cdot \vdash_U C[(s_1 \mid e_1)] : 1 \wedge \cdot \vdash_U C[e_1](s_2 \mid e_2) : 1 \wedge C[(s_1 \mid e_1)] \stackrel{U}{\hookrightarrow}^* \langle\rangle \implies \\ C[(s_2 \mid e_2)] \stackrel{U}{\hookrightarrow}^* \langle\rangle$$

Fig. 4. Contextual Approximation

## 2 PROOFS

Lemma 1.

$$((s_1 \mid v_1),(s_2 \mid v_2)) \in \mathcal{V}[\![\sigma]\!]^j \text{ iff } ((s_1 \mid v_1),(s_2 \mid v_2)) \in \mathcal{E}[\![\sigma]\!]^j$$

Proof. Direct from definition of $\mathcal{E}[\![\sigma]\!]^j$. □

Lemma 2 (Empty Bang Environment Store). *If* $((s_1, s_2) \mid \gamma) \in \mathcal{G}[\![!\Gamma]\!]^j$, *then* $s_1 = s_2 = \emptyset$.

Proof. By induction on $!\Gamma$ and definition of $\mathcal{V}[\![!\sigma]\!]^j$. □

Lemma 3 (Monotonicity).

(1) *If* $(v_1, v_2) \in \mathcal{V}[\![\sigma]\!]^j, j' \le j$ *then* $(v_1, v_2) \in \mathcal{V}[\![\sigma]\!]^{j'}$.
(2) *If* $(e_1, e_2) \in \mathcal{E}[\![\sigma]\!]^j, j' \le j$ *then* $(e_1, e_2) \in \mathcal{E}[\![\sigma]\!]^{j'}$.
(3) *If* $((s_1 \mid v_1),(s_2 \mid v_2)) \in \mathcal{V}[\![\sigma]\!]^j, j' \le j$ *then* $((s_1 \mid v_1),(s_2 \mid v_2)) \in \mathcal{E}[\![\sigma]\!]^{j'}$
(4) *If* $((s_1 \mid e_1),(s_2 \mid e_2)) \in \mathcal{E}[\![\sigma]\!]^j, j' \le j$ *then* $((s_1 \mid e_1),(s_2 \mid e_2)) \in \mathcal{E}[\![\sigma]\!]^{j'}$

Proof. By induction on $\sigma, \sigma$. □

Lemma 4 (Anti-Reduction).

(1) *If* $e_1 \overset{U}{\hookrightarrow}{}^{j'} e'_1, e_2 \overset{U}{\hookrightarrow}{}^{*} e'_2$ *and* $(e'_1, e'_2) \in \mathcal{E}[\![\sigma]\!]^j$, *then* $(e_1, e_2) \in \mathcal{E}[\![\sigma]\!]^{j-j'}$.
(2) *If* $(s_1 \mid e_1) \overset{L}{\hookrightarrow}{}^{j'} (s'_1 \mid e'_1), (s_2 \mid e_2) \overset{L}{\hookrightarrow}{}^{*} (s'_2 \mid e'_2)$ *and* $((s'_1 \mid e'_1),(s'_2 \mid e'_2)) \in \mathcal{E}[\![\sigma]\!]^j$, *then* $((s_1 \mid e_1),(s_2 \mid e_2)) \in \mathcal{E}[\![\sigma]\!]^{j-j'}$.

Proof. Direct from definition of $\mathcal{E}[\![\sigma]\!]^j, \mathcal{E}[\![\sigma]\!]^j$. □

Lemma 5 (Compositionality). *For any closed* $\sigma$

(1) $\mathcal{V}[\![\rho[\sigma/\alpha]]\!]^j = \mathcal{V}[\![\rho[(\mathcal{V}[\![\sigma]\!]^-, \sigma, \sigma)/\alpha]]\!]^j$
(2) $\mathcal{E}[\![\rho[\sigma/\alpha]]\!]^j = \mathcal{E}[\![\rho[(\mathcal{V}[\![\sigma]\!]^-, \sigma, \sigma)/\alpha]]\!]^j$
(3) $\mathcal{V}[\![\rho[\sigma/\alpha]]\!]^j = \mathcal{V}[\![\rho[(\mathcal{V}[\![\sigma]\!]^-, \sigma, \sigma)/\alpha]]\!]^j$
(4) $\mathcal{E}[\![\rho[\sigma/\alpha]]\!]^j = \mathcal{E}[\![\rho[(\mathcal{V}[\![\sigma]\!]^-, \sigma, \sigma)/\alpha]]\!]^j$

### 2.1 Splitting Lemma

Lemma 6 (Splitting and Relational Substitution). *If* $\Gamma = \Gamma_1 \boxplus \Gamma_2$ *and* $\mathrm{dom}(\gamma) = \Gamma, \mathrm{dom}(\gamma_1) = \Gamma_1, \mathrm{dom}(\gamma_2) = \Gamma_2$ *and* $\gamma = \gamma_1 \boxplus \gamma_2$ *then*

(1) *if* $\Gamma_1 \vdash \sigma$, *then* $(\gamma)_R(\sigma) = (\gamma_1)_R(\sigma)$
(2) *if* $\Gamma_1 \vdash \sigma$, *then* $(\gamma)_R(\sigma) = (\gamma_1)_R(\sigma)$
(3) *if* $\Gamma_2 \vdash \sigma$, *then* $(\gamma)_R(\sigma) = (\gamma_2)_R(\sigma)$
(4) *if* $\Gamma_2 \vdash \sigma$, *then* $(\gamma)_R(\sigma) = (\gamma_2)_R(\sigma)$

Proof. Without loss of generality, consider the first case. Since $\gamma = \gamma_1 \boxplus \gamma_2$ and $\Gamma_1 \vdash \sigma$, every free type variable $\alpha \in \sigma$ we have $\alpha \in \Gamma_1$, and since $\gamma = \gamma_1 \boxplus \gamma_2$, we have $(\gamma)_R(\alpha) = (\gamma_1)_R(\alpha)$, thus $(\gamma)_R(\sigma) = (\gamma_1)_R(\sigma)$. □

Lemma 7 (Splitting Lemma). *If* $\Gamma = \Gamma' \boxplus \Gamma''$, *and* $((s_1, s_2) \mid \gamma) \in \mathcal{G}[\![\Gamma]\!]^j$, *then there exist* $s'_1, s'_2, \gamma', s''_1, s''_2, \gamma''$ *such that* $s_1 = s'_1 + s''_1, s_2 = s'_2 + s''_2, \gamma = \gamma' \boxplus \gamma''$, $((s'_1, s'_2) \mid \gamma') \in \mathcal{G}[\![\Gamma']\!]^j$, *and* $((s''_1, s''_2) \mid \gamma'') \in \mathcal{G}[\![\Gamma'']\!]^j$.

Proof. By induction on $\Gamma', \Gamma''$. Without loss of generality, we only consider cases where non-shared variables are in $\Gamma'$.

**Case** $\Gamma = (\Gamma', x{:}!\sigma) \boxplus (\Gamma'', x{:}!\sigma) = (\Gamma' \boxplus \Gamma''), x{:}!\sigma$:

Then by inductive hypothesis we have

$$((s_1, s_2) \mid \gamma) = ((s_1' + s_1'' + s_1''', s_2' + s_2'' + s_2''') \mid \gamma'''[x \mapsto (v_1, v_2)]),$$

where $\gamma''' = \gamma' \boxplus \gamma''$, $((s_1', s_2') \mid \gamma') \in \mathcal{G}\,[\![\Gamma']\!]^j$, $((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$ and $((s_1''' \mid v_1), (s_2''' \mid v_2)) \in \mathcal{V}\,[\![!(\gamma''')_R(\sigma)]\!]^j$.

By definition of $\mathcal{V}\,[\![!(\gamma''')_R(\sigma)]\!]^j$, we have $s_1''' = s_2''' = \emptyset$, so $s_1 = s_1' + s_1''$, $s_2 = s_2' + s_2''$. Then we can show

$$((s_1', s_2') \mid \gamma'[x \mapsto (v_1, v_2)]) \in \mathcal{G}\,[\![\Gamma', x{:}!\sigma]\!]^j \qquad ((s_1'', s_2'') \mid \gamma''[x \mapsto (v_1, v_2)]) \in \mathcal{G}\,[\![\Gamma'', x{:}!\sigma]\!]^j$$

by Lemma 6 (Splitting and Relational Substitution). We conclude by verifying that

$$\gamma'''[x \mapsto (v_1, v_2)] = \gamma'[x \mapsto (v_1, v_2)] \boxplus \gamma''[x \mapsto (v_1, v_2)]$$

**Case** $\Gamma = (\Gamma', x{:}\sigma) \boxplus \Gamma'' = (\Gamma' \boxplus \Gamma''), x{:}\sigma$, with $x \notin \Gamma''$:

By inductive hypothesis we have

$$((s_1, s_2) \mid \gamma) = ((s_1' + s_1'' + s_{1,x}, s_2' + s_2'' + s_{1,x}) \mid \gamma'''[x \mapsto (v_1, v_2)]),$$

where $\gamma''' = \gamma' \boxplus \gamma''$, $((s_1', s_2') \mid \gamma') \in \mathcal{G}\,[\![\Gamma']\!]^j$, $((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$ and $((s_{1,x} \mid v_1), (s_{2,x} \mid v_2)) \in \mathcal{V}\,[\![(\gamma''')_R(\sigma)]\!]^j$.

So we have

$$((s_1' + s_{1,x}, s_2' + s_{2,x}) \mid \gamma'[x \mapsto (v_1, v_2)]) \in \mathcal{G}\,[\![\Gamma', x{:}\sigma]\!]^j \qquad ((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$$

the latter by inductive hypothesis, and the former by Lemma 6 (Splitting and Relational Substitution).

Finally, we verify that

$$\gamma'''[x \mapsto (v_1, v_2)] = \gamma'[x \mapsto (v_1, v_2)] \boxplus \gamma''$$

**Case** $\Gamma = (\Gamma', x{:}\sigma) \boxplus (\Gamma'', x{:}\sigma) = (\Gamma' \boxplus \Gamma''), x{:}\sigma$:

By inductive hypothesis we have

$$((s_1, s_2) \mid \gamma) = ((s_1' + s_1'', s_2' + s_2'') \mid \gamma'''[x \mapsto (v_1, v_2)]),$$

where $\gamma''' = \gamma' \boxplus \gamma''$, $((s_1', s_2') \mid \gamma') \in \mathcal{G}\,[\![\Gamma']\!]^j$, $((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$ and $(v_1, v_2) \in \mathcal{V}\,[\![(\gamma''')_R(\sigma)]\!]^j$. So we have

$$((s_1', s_2') \mid \gamma'[x \mapsto (v_1, v_2)]) \in \mathcal{G}\,[\![\Gamma', x{:}\sigma]\!]^j \qquad ((s_1'', s_2'') \mid \gamma''[x \mapsto (v_1, v_2)]) \in \mathcal{G}\,[\![\Gamma'', x{:}\sigma]\!]^j$$

By Lemma 6 (Splitting and Relational Substitution).

And we conclude by verifying

$$\gamma'''[x \mapsto (v_1, v_2)] = \gamma'[x \mapsto (v_1, v_2)] \boxplus \gamma''[x \mapsto (v_1, v_2)]$$

**Case** $\Gamma = (\Gamma', x{:}\sigma) \boxplus \Gamma'' = (\Gamma' \boxplus \Gamma''), x{:}\sigma$ with $x \notin \Gamma''$: By inductive hypothesis we have

$$((s_1, s_2) \mid \gamma) = ((s_1' + s_1'', s_2' + s_2'') \mid \gamma'''[x \mapsto (v_1, v_2)]),$$

where $\gamma''' = \gamma' \boxplus \gamma''$, $((s_1', s_2') \mid \gamma') \in \mathcal{G}\,[\![\Gamma']\!]^j$, $((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$ and $(v_1, v_2) \in \mathcal{V}\,[\![(\gamma''')_R(\sigma)]\!]^j$. So we have

$$((s_1', s_2') \mid \gamma'[x \mapsto (v_1, v_2)]) \in \mathcal{G}\,[\![\Gamma', x{:}\sigma]\!]^j \qquad ((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$$

The latter by assumption and the former by Lemma 6 (Splitting and Relational Substitution).
And finally we verify

$$\gamma'''[x \mapsto (v_1, v_2)] = \gamma'[x \mapsto (v_1, v_2)] \boxplus \gamma''$$

**Case** $\Gamma = (\Gamma', \alpha) \boxplus (\Gamma'', \alpha) = (\Gamma' \boxplus \Gamma''), \alpha$: By inductive hypothesis we have

$$((s_1, s_2) \mid \gamma) = ((s_1' + s_1'', s_2' + s_2'') \mid \gamma'''[\alpha \mapsto (R, \sigma_1, \sigma_2)]),$$

where $\gamma''' = \gamma' \boxplus \gamma''$, $((s_1', s_2') \mid \gamma') \in \mathcal{G}\,[\![\Gamma']\!]^j$, $((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$ and $R \in \mathrm{Rel}[\sigma_1, \sigma_2]$, so

$$((s_1', s_2') \mid \gamma'[\alpha \mapsto (R, \sigma_1, \sigma_2)]) \in \mathcal{G}\,[\![\Gamma', \alpha]\!]^j \qquad ((s_1'', s_2'') \mid \gamma''[\alpha \mapsto (R, \sigma_1, \sigma_2)]) \in \mathcal{G}\,[\![\Gamma'', \alpha]\!]^j$$
$$\gamma'''[\alpha \mapsto (R, \sigma_1, \sigma_2)] = \gamma'[\alpha \mapsto (R, \sigma_1, \sigma_2)] \boxplus \gamma''[\alpha \mapsto (R, \sigma_1, \sigma_2)]$$

**Case** $\Gamma = (\Gamma', \alpha) \boxplus \Gamma'' = (\Gamma' \boxplus \Gamma''), \alpha$ with $\alpha \notin \Gamma''$: By inductive hypothesis we have

$$((s_1, s_2) \mid \gamma) = ((s_1' + s_1'', s_2' + s_2'') \mid \gamma'''[\alpha \mapsto (R, \sigma_1, \sigma_2)]),$$

where $\gamma''' = \gamma' \boxplus \gamma''$, $((s_1', s_2') \mid \gamma') \in \mathcal{G}\,[\![\Gamma']\!]^j$, $((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$ and $R \in \mathrm{Rel}[\sigma_1, \sigma_2]$, so

$$((s_1', s_2') \mid \gamma'[\alpha \mapsto (R, \sigma_1, \sigma_2)]) \in \mathcal{G}\,[\![\Gamma', \alpha]\!]^j \qquad ((s_1'', s_2'') \mid \gamma'') \in \mathcal{G}\,[\![\Gamma'']\!]^j$$
$$\gamma'''[\alpha \mapsto (R, \sigma_1, \sigma_2)] = \gamma'[\alpha \mapsto (R, \sigma_1, \sigma_2)] \boxplus \gamma''$$

□

### 2.2 Monadic Bind

Lemma 8 (Monadic Bind). *If* $((s_1 \mid e_1), (s_2 \mid e_2)) \in \mathcal{E}\,[\![\rho]\!]^j$, $s_3 = s_1 + s_1'$, $s_4 = s_2 + s_2'$, *and*

$$\begin{gathered}\forall j'' \leq j, ((s_1'' \mid v_1''), (s_2'' \mid v_2'')) \in \mathcal{V}\,[\![\rho]\!]^{j''}, s_3', s_4'.\\ s_3' = s_1' + s_1'' \wedge s_4' = s_2' + s_2'' \Rightarrow\\ ((s_3' \mid K_1[v_1]), (s_4' \mid K_2[v_2])) \in \mathcal{E}\,[\![\rho']\!]^{j''+j'}\end{gathered}$$

*then*

$$((s_3 \mid K_1[e_1]), (s_4 \mid K_2[e_1])) \in \mathcal{E}\,[\![\rho']\!]^{j+j'}$$

Proof. Consider $j'' \leq j + j'$ and $(s_3' \mid v_1'')$ such that

$$(s_3 \mid K_1[e_1]) \overset{L}{\hookrightarrow}{}^{j''} (s_3' \mid v_1'') \tag{1}$$

We need to show

$$\exists (s_4' \mid v_2'').(s_4 \mid K_2[e_2]) \overset{L}{\hookrightarrow}^* (s_4' \mid v_2'') \wedge ((s_3' \mid v_1''), (s_4' \mid v_2'')) \in \mathcal{V}\,[\![\rho']\!]^{j+j'-j''}$$

Because of (1), there must exist some $j''' \leq j''$ and $(s_1'' \mid v_1)$ such that

$$(s_1 \mid e_1) \overset{L}{\hookrightarrow}^{j'''} (s_1'' \mid v_1) \tag{2}$$

We assumed $((s_1 \mid e_1), (s_2 \mid e_2)) \in \mathcal{E}\,[\![\rho]\!]^j$. Instantiate this with $j'''$ and $(s_1'' \mid v_1)$ to get that there exists $(s_2'' \mid v_2)$ such that $(s_2 \mid e_2) \overset{L}{\hookrightarrow}^* (s_2'' \mid v_2)$ and

$$((s_1'' \mid v_1), (s_2'' \mid v_2)) \in \mathcal{V}\,[\![\rho]\!]^{j-j'''} \tag{3}$$

Next, instantiate our second premise with (3) to find

$$((s_1'' + s_1' \mid K_1[v_1]), (s_2'' + s_2' \mid K_2[v_2])) \in \mathcal{E}\,[\![\rho']\!]^{j-j'''+j'} \tag{4}$$

From (1) and (2) and we deduce $(s_1'' + s_1' \mid K_1[v_1]) \overset{L}{\hookrightarrow}^{j''-j'''} (s_3' \mid v_1'')$. Instantiate (4) with this to find there exists $(s_4' \mid v_2'')$, such that

$$(s_2'' + s_2' \mid K_2[v_2]) \overset{L}{\hookrightarrow}^* (s_4' \mid v_2'') \tag{5}$$

$$((s_3' \mid v_1''), (s_4' \mid v_2'')) \in \mathcal{V}\,[\![\rho']\!]^{j+j'-j''} \tag{6}$$

All that remains is to show is $(s_2 + s_2' \mid K_2[e_2]) \overset{L}{\hookrightarrow}^* (s_4' \mid v_2'')$. Since $(s_2 \mid e_2) \overset{L}{\hookrightarrow}^* (s_2'' \mid v_2)$, the operational semantics give us $(s_2 + s_2' \mid K_2[e_2]) \overset{L}{\hookrightarrow}^* (s_2'' + s_2' \mid K_2[v_2])$. We have the rest from (5).

□

Lemma 9 (Monadic Bind Under Share).

(1) *If* $((s_1 \mid e_1), (s_2 \mid e_2)) \in \mathcal{E}\,[\![\rho]\!]^j$, $s_4 = s_2 + s_2'$, $\Psi_1; \cdot \vdash_L s_1 \mid e_1 : (\rho)_1$, $\Psi_2; \cdot \vdash_L s_2 \mid e_2 : (\rho)_2$, *and*

$$\begin{aligned}&\forall j'' \leq j, ((s_1'' \mid v_1''), (s_2'' \mid v_2'')) \in \mathcal{V}\,[\![\rho]\!]^{j''}, s_4', \Psi_1'', \Psi_2''.\\ &\quad s_4' = s_2' + s_2'' \wedge\\ &\quad \Psi_1''; \cdot \vdash_L s_1'' \mid v_1'' : (\rho)_1 \wedge \Psi_2''; \cdot \vdash_L s_2'' \mid v_2'' : (\rho)_2 \Rightarrow\\ &\quad ((s_1' \mid K_1[\mathsf{share}(s_1'' : \Psi_1'').\, v_1]), (s_4' \mid K_2[v_2])) \in \mathcal{E}\,[\![\rho']\!]^{j''+j'}\end{aligned}$$

*then*

$$((s_1' \mid K_1[\mathsf{share}(s_1 : \Psi_1).\, e_1]), (s_4 \mid K_2[e_2])) \in \mathcal{E}\,[\![\rho']\!]^{j+j'}$$

(2) *If* $((s_1 \mid e_1), (s_2 \mid e_2)) \in \mathcal{E}\,[\![\rho]\!]^j$, $s_3 = s_1 + s_1'$, $\Psi_1; \cdot \vdash_L s_1 \mid e_1 : (\rho)_1$, $\Psi_2; \cdot \vdash_L s_2 \mid e_2 : (\rho)_2$, *and*

$$\begin{aligned}&\forall j'' \leq j, ((s_1'' \mid v_1''), (s_2'' \mid v_2'')) \in \mathcal{V}\,[\![\rho]\!]^{j''}, s_3', \Psi_1'', \Psi_2''.\\ &\quad s_3' = s_1' + s_1'' \wedge\\ &\quad \Psi_1''; \cdot \vdash_L s_1'' \mid v_1'' : (\rho)_1 \wedge \Psi_2''; \cdot \vdash_L s_2'' \mid v_2'' : (\rho)_2 \Rightarrow\\ &\quad ((s_3' \mid K_1[v_1]), (s_2' \mid K_2[\mathsf{share}(s_2'' : \Psi_2'').\, v_2])) \in \mathcal{E}\,[\![\rho']\!]^{j''+j'}\end{aligned}$$

*then*

$$((s_3 \mid K_1[e_1]), (s_2' \mid K_2[\mathsf{share}(s_2 : \Psi_2).\, e_2])) \in \mathcal{E}\,[\![\rho']\!]^{j+j'}$$

(3) *If* $((s_1 \mid e_1), (s_2 \mid e_2)) \in \mathcal{E}[\![\rho]\!]^j$, $\Psi_1; \cdot \vdash_L s_1 \mid e_1 : (\rho)_1$, $\Psi_2; \cdot \vdash_L s_2 \mid e_2 : (\rho)_2$, *and*

$$\begin{aligned}&\forall j'' \leq j, ((s_1'' \mid v_1''), (s_2'' \mid v_2'')) \in \mathcal{V}[\![\rho]\!]^{j''}, \Psi_1'', \Psi_2''.\\ &\quad \Psi_1''; \cdot \vdash_L s_1'' \mid v_1'' : (\rho)_1 \land \Psi_2''; \cdot \vdash_L s_2'' \mid v_2'' : (\rho)_2 \Rightarrow\\ &\quad ((s_1' \mid K_1[\mathsf{share}(s_1'' : \Psi_1'').\, v_1]), (s_2' \mid K_2[\mathsf{share}(s_2'' : \Psi_2'').\, v_2])) \in \mathcal{E}[\![\rho']\!]^{j''+j'}\end{aligned}$$

*then*

$$((s_1' \mid K_1[\mathsf{share}(s_1 : \Psi_1).\, e_1]), (s_2' \mid K_2[\mathsf{share}(s_2 : \Psi_2).\, e_2])) \in \mathcal{E}[\![\rho']\!]^{j+j'}$$

Proof. All parts are similar to Lemma 8 (Monadic Bind). □

## 2.3 Copy Lemma

Lemma 10 (Copy). *If* $((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\sigma]\!]^j$ *then*

$$((\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_1 : \Psi_1).\, v_1), (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_2 : \Psi_2).\, v_2)) \in \mathcal{E}[\![\sigma]\!]^j$$

Proof. By induction on $\sigma$.

**Case** $1$

We know $s_i = \emptyset$ and $v_i = \langle\rangle$. Note that $(\emptyset \mid \mathsf{copy}^1\ \mathsf{share}(\emptyset : \cdot).\, \langle\rangle) \overset{L}{\hookrightarrow} (\emptyset \mid \langle\rangle)$. By closure under anti-reduction, it suffices to show

$$((\emptyset \mid \langle\rangle), (\emptyset \mid \langle\rangle)) \in \mathcal{E}[\![\sigma]\!]^j \supseteq \mathcal{V}[\![\sigma]\!]^j$$

Which is immediate.

**Case** $\sigma' \otimes \sigma''$

We assumed $((s_1' + s_1'' \mid \langle v_1', v_1'' \rangle), (s_2' + s_2'' \mid \langle v_2', v_2'' \rangle)) \in \mathcal{V}[\![\sigma' \otimes \sigma'']\!]^j$. Therefore,

$$((s_1' \mid v_1'), (s_2' \mid v_2')) \in \mathcal{V}[\![\sigma']\!]^j \tag{7}$$

$$((s_1'' \mid v_1''), (s_2'' \mid v_2'')) \in \mathcal{V}[\![\sigma'']\!]^j \tag{8}$$

We need to show

$$((\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_1' + s_1'' : \Psi_1).\, \langle v_1', v_1'' \rangle), (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_2' + s_2'' : \Psi_2).\, \langle v_2', v_2'' \rangle)) \in \mathcal{E}[\![\sigma' \otimes \sigma'']\!]^j$$

Note

$$(\emptyset \mid \mathsf{copy}^{\sigma' \otimes \sigma''}\ \mathsf{share}(s_i' + s_i'' : \Psi_i).\, \langle v_i', v_i'' \rangle) \overset{L}{\hookrightarrow} (\emptyset \mid \langle \mathsf{copy}^{\sigma'}\ \mathsf{share}(s_i' : \Psi_i').\, v_i', \mathsf{copy}^{\sigma''}\ \mathsf{share}(s_i'' : \Psi_i'').\, v_i'' \rangle)$$

By closure under anti-reduction, it suffices to show

$$\begin{aligned}&((\emptyset \mid \langle \mathsf{copy}^{\sigma'}\ \mathsf{share}(s_1' : \Psi_1').\, v_1', \mathsf{copy}^{\sigma''}\ \mathsf{share}(s_1'' : \Psi_1'').\, v_1'' \rangle),\\ &\ (\emptyset \mid \langle \mathsf{copy}^{\sigma'}\ \mathsf{share}(s_2' : \Psi_2').\, v_2', \mathsf{copy}^{\sigma''}\ \mathsf{share}(s_2'' : \Psi_2'').\, v_2'' \rangle)) \in \mathcal{E}[\![\sigma' \otimes \sigma'']\!]^j\end{aligned}$$

From (7), (8), and the induction hypothesis,

$$((\emptyset \mid \mathsf{copy}^{\sigma'}\ \mathsf{share}(s_1 : \Psi_1').\, v_1'), (\emptyset \mid \mathsf{copy}^{\sigma'}\ \mathsf{share}(s_2 : \Psi_2').\, v_2')) \in \mathcal{E}[\![\sigma']\!]^j$$

$$((\emptyset \mid \mathsf{copy}^{\sigma''}\ \mathsf{share}(s_1'' : \Psi_1'').\, v_1''), (\emptyset \mid \mathsf{copy}^{\sigma''}\ \mathsf{share}(s_2'' : \Psi_2'').\, v_2'')) \in \mathcal{E}[\![\sigma'']\!]^j$$

Applying monadic bind twice, assume $j' \leq j$ and

$$((\mathsf{s}_3' \mid \mathsf{v}_3'), (\mathsf{s}_4' \mid \mathsf{v}_4')) \in \mathcal{V}\llbracket \sigma' \rrbracket^{j'} \tag{9}$$

$$((\mathsf{s}_3'' \mid \mathsf{v}_3''), (\mathsf{s}_4'' \mid \mathsf{v}_4'')) \in \mathcal{V}\llbracket \sigma'' \rrbracket^{j'} \tag{10}$$

It suffices to show

$$((\mathsf{s}_3' + \mathsf{s}_3'' \mid \langle \mathsf{v}_3', \mathsf{v}_3'' \rangle), (\mathsf{s}_4' + \mathsf{s}_4'' \mid \langle \mathsf{v}_4', \mathsf{v}_4'' \rangle)) \in \mathcal{E}\llbracket \sigma' \otimes \sigma'' \rrbracket^{j'} \supseteq \mathcal{V}\llbracket \sigma' \otimes \sigma'' \rrbracket^{j'}$$

Which follows from (9) and (10).

**Case** $\sigma_1 \oplus \sigma_2$

We assumed

$$((\mathsf{s}_1 \mid \mathsf{inj_n}\ \mathsf{v}_1'), (\mathsf{s}_2 \mid \mathsf{inj_n}\ \mathsf{v}_2')) \in \mathcal{V}\llbracket \sigma_1 \oplus \sigma_2 \rrbracket^{j}$$

Therefore,

$$((\mathsf{s}_1 \mid \mathsf{v}_1'), (\mathsf{s}_2 \mid \mathsf{v}_2')) \in \mathcal{V}\llbracket \sigma_n \rrbracket^{j} \tag{11}$$

We need to show

$$((\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}_1 : \Psi_1).\ \mathsf{inj_n}\ \mathsf{v}_1'), (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}_2 : \Psi_2).\ \mathsf{inj_n}\ \mathsf{v}_2')) \in \mathcal{E}\llbracket \sigma_1 \oplus \sigma_2 \rrbracket^{j}$$

Note $\mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}_i : \Psi_i).\ \mathsf{inj_n}\ \mathsf{v}_i' \overset{\mathsf{L}}{\hookrightarrow} \mathsf{inj_n}\ \mathsf{copy}^{\sigma_n}\ \mathsf{share}(\mathsf{s}_i : \Psi_i).\ \mathsf{v}_i'$. By closure under anti-reduction, it suffices to show

$$((\emptyset \mid \mathsf{inj_n}\ \mathsf{copy}^{\sigma_n}\ \mathsf{share}(\mathsf{s}_1 : \Psi_1).\ \mathsf{v}_1'), (\emptyset \mid \mathsf{inj_n}\ \mathsf{copy}^{\sigma_n}\ \mathsf{share}(\mathsf{s}_2 : \Psi_2).\ \mathsf{v}_2')) \in \mathcal{E}\llbracket \sigma_1 \oplus \sigma_2 \rrbracket^{j}$$

From (11) and the induction hypothesis,

$$((\emptyset \mid \mathsf{copy}^{\sigma_n}\ \mathsf{share}(\mathsf{s}_1 : \Psi_1).\ \mathsf{v}_1'), (\emptyset \mid \mathsf{copy}^{\sigma_n}\ \mathsf{share}(\mathsf{s}_2 : \Psi_2).\ \mathsf{v}_2')) \in \mathcal{E}\llbracket \sigma_n \rrbracket^{j}$$

Assume $j' \leq j$ and

$$((\mathsf{s}_1' \mid \mathsf{v}_1''), (\mathsf{s}_2' \mid \mathsf{v}_2'')) \in \mathcal{V}\llbracket \sigma_n \rrbracket^{j'} \tag{12}$$

By monadic bind, it suffices to show

$$((\mathsf{s}_1' \mid \mathsf{inj_n}\ \mathsf{v}_1''), (\mathsf{s}_2' \mid \mathsf{inj_n}\ \mathsf{v}_2'')) \in \mathcal{E}\llbracket \sigma_1 \oplus \sigma_2 \rrbracket^{j'} \supseteq \mathcal{V}\llbracket \sigma_1 \oplus \sigma_2 \rrbracket^{j'}$$

Which follows from (12).

**Case** $\mu\alpha.\ \sigma$

We assumed

$$((\mathsf{s}_1 \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_1'), (\mathsf{s}_2 \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_2')) \in \mathcal{V}\llbracket \mu\alpha.\ \sigma \rrbracket^{j}$$

Therefore,

$$((\mathsf{s}_1 \mid \mathsf{v}_1'), (\mathsf{s}_2 \mid \mathsf{v}_2')) \in \mathcal{V}\llbracket \sigma[\mu\alpha.\ \sigma/\alpha] \rrbracket^{j-1} \tag{13}$$

We need to show

$$((\emptyset \mid \mathsf{copy}^{\mu\alpha.\sigma}\ \mathsf{share}(\mathsf{s}_1 : \Psi_1).\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_1'), (\emptyset \mid \mathsf{copy}^{\mu\alpha.\sigma}\ \mathsf{share}(\mathsf{s}_2 : \Psi_2).\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_2')) \in \mathcal{E}\llbracket \mu\alpha.\ \sigma \rrbracket^{j}$$

Note $\mathsf{copy}^{\mu\alpha.\sigma}\ \mathsf{share}(\mathsf{s}_i : \Psi_i).\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}'_i \overset{\mathsf{L}}{\hookrightarrow} \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{copy}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{share}(\mathsf{s}_i : \Psi_i).\ \mathsf{v}'_i$. By closure under anti-reduction, it suffices to show

$$((\emptyset \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{copy}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{share}(\mathsf{s}_1 : \Psi_1).\ \mathsf{v}'_1), (\emptyset \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{copy}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{share}(\mathsf{s}_2 : \Psi_2).\ \mathsf{v}'_2)) \in \mathcal{E}\,[\![\mu\alpha.\,\sigma]\!]^j$$

By (13) and the induction hypothesis,

$$((\emptyset \mid \mathsf{copy}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{share}(\mathsf{s}_1 : \Psi_1).\ \mathsf{v}'_1), (\emptyset \mid \mathsf{copy}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{share}(\mathsf{s}_2 : \Psi_2).\ \mathsf{v}'_2)) \in \mathcal{E}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j-1}$$

Assume $j' \leq j - 1$ and

$$((\mathsf{s}'_1 \mid \mathsf{v}''_1), (\mathsf{s}'_2 \mid \mathsf{v}''_2)) \in \mathcal{V}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j'} \tag{14}$$

By monadic bind, it suffices to show

$$((\mathsf{s}'_1 \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}''_1), (\mathsf{s}'_2 \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}''_2)) \in \mathcal{E}\,[\![\mu\alpha.\,\sigma]\!]^{j'+1} \supseteq \mathcal{V}\,[\![\mu\alpha.\,\sigma]\!]^{j'+1}$$

Which follows from (14) and downward closure.

**Case** $!\sigma$

We assumed

$$((\mathsf{s}_1 \mid \mathsf{share}(\mathsf{s}'_1 : \Psi'_1).\ \mathsf{v}'_1), (\mathsf{s}_2 \mid \mathsf{share}(\mathsf{s}'_2 : \Psi'_2).\ \mathsf{v}'_2)) \in \mathcal{V}\,[\![!\sigma]\!]^j \tag{15}$$

Therefore, $\mathsf{s}_1 = \mathsf{s}_2 = \emptyset$, $\Psi_1 = \Psi_2 = \cdot$. We need to show

$$((\emptyset \mid \mathsf{copy}^{!\sigma}\ \mathsf{share}(\emptyset : \cdot).\ \mathsf{share}(\mathsf{s}'_1 : \Psi_1).\ \mathsf{v}'_1), (\emptyset \mid \mathsf{copy}^{!\sigma}\ \mathsf{share}(\emptyset : \cdot).\ \mathsf{share}(\mathsf{s}'_2 : \Psi_2).\ \mathsf{v}'_2)) \in \mathcal{E}\,[\![\mu\alpha.\,\sigma]\!]^j$$

Note $\mathsf{copy}^{!\sigma}\ \mathsf{share}(\emptyset : \cdot).\ \mathsf{share}(\mathsf{s}'_i : \Psi_i).\ \mathsf{v}'_i \overset{\mathsf{L}}{\hookrightarrow} \mathsf{share}(\mathsf{s}'_i : \Psi_i).\ \mathsf{v}'_i$. By closure under anti-reduction, it suffices to show

$$((\emptyset \mid \mathsf{share}(\mathsf{s}'_1 : \Psi_1).\ \mathsf{v}'_1), (\emptyset \mid \mathsf{share}(\mathsf{s}'_2 : \Psi_2).\ \mathsf{v}'_2)) \in \mathcal{E}\,[\![!\sigma]\!]^j$$

Since $\mathcal{E}\,[\![!\sigma]\!]^j \supseteq \mathcal{V}\,[\![!\sigma]\!]^j$, we need only show (15).

**Case** $\mathsf{Box}\ 0$

We assumed $((\mathsf{s}_1 \mid \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{v}_2)) \in \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$. Therefore $\mathsf{s}_i = [\ell_i \mapsto \cdot]$ and $\mathsf{v}_i = \ell_i$. We need to show

$$((\emptyset \mid \mathsf{copy}^{\mathsf{Box}\ 0}\ \mathsf{share}([\ell_1 \mapsto \cdot] : (\cdot; \cdot \vdash \ell : \mathsf{Box}\ 0)).\ \ell_1),$$
$$(\emptyset \mid \mathsf{copy}^{\mathsf{Box}\ 0}\ \mathsf{share}([\ell_2 \mapsto \cdot] : (\cdot; \cdot \vdash \ell : \mathsf{Box}\ 0)).\ \ell_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j$$

Note $\mathsf{copy}^{!\sigma}\ \mathsf{share}([\ell_i \mapsto \cdot] : ).\ \ell_i \overset{\mathsf{L}}{\hookrightarrow} ([\ell'_i \mapsto \cdot] \mid \ell'_i)$. By closure under anti-reduction, it suffices to show

$$(([\ell'_1 \mapsto \cdot] \mid \ell'_1), ([\ell'_2 \mapsto \cdot] \mid \ell'_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j$$

Since $\mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j \supseteq \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$, we need only show

$$(([\ell'_1 \mapsto \cdot] \mid \ell'_1), ([\ell'_2 \mapsto \cdot] \mid \ell'_2)) \in \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$$

Which is immediate from the definition of $\mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$.

**Case** $\mathsf{Box}\ 1\ \sigma$

We assumed

$$(([\ell_1 \mapsto (\mathsf{s}'_1 \mid \mathsf{v}'_1)] \mid \ell_1), ([\ell_2 \mapsto (\mathsf{s}'_2 \mid \mathsf{v}'_2)] \mid \ell_2)) \in \mathcal{V}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^j$$

Therefore $\Psi_i = (\Psi'_i; \cdot \vdash \ell_i : \mathsf{Box}\ 1\ \sigma)$ and

$$((\mathsf{s}'_1 \mid \mathsf{v}'_1), (\mathsf{s}'_2 \mid \mathsf{v}'_2)) \in \mathcal{V}\,[\![\sigma]\!]^j \tag{16}$$

We need to show

$$((\emptyset \mid \mathsf{copy}^{\mathsf{Box}\ 1\ \sigma}\ \mathsf{share}([\ell_1 \mapsto (\mathsf{s}'_1 \mid \mathsf{v}'_1)] : (\Psi'_1; \cdot \vdash \ell_1 : \mathsf{Box}\ 1\ \sigma)).\ \ell_1),$$
$$(\emptyset \mid \mathsf{copy}^{\mathsf{Box}\ 1\ \sigma}\ \mathsf{share}([\ell_2 \mapsto (\mathsf{s}'_2 \mid \mathsf{v}'_2)] : (\Psi'_2; \cdot \vdash \ell_2 : \mathsf{Box}\ 1\ \sigma)).\ \ell_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^j$$

Note

$$(\emptyset \mid \mathsf{copy}^{\mathsf{Box}\ 1\ \sigma}\ \mathsf{share}([\ell_i \mapsto (\mathsf{s}'_i \mid \mathsf{v}'_i)] : (\Psi'_i; \cdot \vdash \ell_i : \mathsf{Box}\ 1\ \sigma)).\ \ell_i) \stackrel{\mathsf{L}}{\hookrightarrow}$$
$$(\emptyset \mid \mathsf{box}\ \langle \mathsf{new}\ \langle\rangle, \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}'_i : \Psi'_i).\ \mathsf{v}'_i\rangle) \stackrel{\mathsf{L}}{\hookrightarrow}$$
$$([\ell'_i \mapsto \cdot] \mid \mathsf{box}\ \langle \ell'_i, \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}'_i : \Psi'_i).\ \mathsf{v}'_i\rangle)$$

By closure under anti-reduction, it suffices to show

$$(([\ell'_1 \mapsto \cdot] \mid \mathsf{box}\ \langle \ell'_1, \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}'_1 : \Psi'_1).\ \mathsf{v}'_1\rangle),$$
$$([\ell'_2 \mapsto \cdot] \mid \mathsf{box}\ \langle \ell'_2, \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}'_2 : \Psi'_2).\ \mathsf{v}'_2\rangle)) \in \mathcal{E}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^j$$

By (16) and the induction hypothesis,

$$((\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}'_1 : \Psi'_1).\ \mathsf{v}'_1), (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(\mathsf{s}'_2 : \Psi'_2).\ \mathsf{v}'_2)) \in \mathcal{E}\,[\![\sigma]\!]^j$$

Assume $j' \le j$ and

$$((\mathsf{s}''_1 \mid \mathsf{v}''_1), (\mathsf{s}''_2 \mid \mathsf{v}''_2)) \in \mathcal{V}\,[\![\sigma]\!]^{j'} \tag{17}$$

By monadic bind, it suffices to show

$$((\mathsf{s}''_1[\ell'_1 \mapsto \cdot] \mid \mathsf{box}\ \langle \ell'_1, \mathsf{v}''_1\rangle), (\mathsf{s}''_2[\ell'_2 \mapsto \cdot] \mid \mathsf{box}\ \langle \ell'_2, \mathsf{v}''_2\rangle)) \in \mathcal{E}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^{j'}$$

By closure under anti-reduction again, we need only show

$$(([\ell'_1 \mapsto (\mathsf{s}''_1 \mid \mathsf{v}''_1)] \mid \ell'_1), ([\ell'_2 \mapsto (\mathsf{s}''_2 \mid \mathsf{v}''_2)] \mid \ell'_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^{j'}$$

Since $\mathcal{E}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^{j'} \supseteq \mathcal{V}\,[\![\mathsf{Box}\ 1\ \sigma]\!]^{j'}$, it suffices to show (17).

□

### 2.4 Compatibility

Lemma 11 (Compat Var).

$$!\Gamma, \mathsf{x}{:}\sigma \vdash_L (\emptyset \mid \mathsf{x}) \precsim^{log} (\emptyset \mid \mathsf{x}) : \sigma$$

Proof. Assume $j \ge 0$ and

$$((\mathsf{s}_1, \mathsf{s}_2) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma, \mathsf{x}{:}\sigma]\!]^j \tag{18}$$

We need to show $((\mathsf{s}_1 \mid (\gamma)_1(\mathsf{x})), (\mathsf{s}_2 \mid (\gamma)_2(\mathsf{x}))) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^j \supseteq \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^j$. All variables of ! type must be mapped to configurations with empty stores, so from (18) we have $((\mathsf{s}_1 \mid (\gamma)_1(\mathsf{x})), (\mathsf{s}_2 \mid (\gamma)_2(\mathsf{x}))) \in \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^j$. □

Lemma 12 (Compat Lambda).

$$\frac{\Gamma, \mathsf{x}{:}\sigma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : \sigma'}{\Gamma \vdash_L (\mathsf{s}_1 \mid \lambda(\mathsf{x}:\sigma).\, \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \lambda(\mathsf{x}:\sigma).\, \mathsf{e}_2) : \sigma \multimap \sigma'}$$

Proof. Assume

$$\Gamma, \mathsf{x}{:}\sigma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : \sigma' \tag{19}$$

Consider $j \geq 0$ and

$$((\mathsf{s}_1', \mathsf{s}_2') \mid \gamma) \in \mathcal{G}\,[\![\Gamma]\!]^j \tag{20}$$

We need to show $((\mathsf{s}_1 + \mathsf{s}_1' \mid \lambda(\mathsf{x}:(\gamma)_1(\sigma)).\,(\gamma)_1(\mathsf{e}_1)), (\mathsf{s}_2 + \mathsf{s}_2' \mid \lambda(\mathsf{x}:(\gamma)_2(\sigma)).\,(\gamma)_2(\mathsf{e}_2))) \in \mathcal{E}\,[\![(\gamma)_R(\sigma) \multimap (\gamma)_R(\sigma')]\!]^j$. Since $\mathcal{E}\,[\![(\gamma)_R(\sigma) \multimap (\gamma)_R(\sigma')]\!]^j \supseteq \mathcal{V}\,[\![(\gamma)_R(\sigma) \multimap (\gamma)_R(\sigma')]\!]^j$, it suffices to show

$$((\mathsf{s}_1 + \mathsf{s}_1' \mid \lambda(\mathsf{x}:(\gamma)_1(\sigma)).\,(\gamma)_1(\mathsf{e}_1)), (\mathsf{s}_2 + \mathsf{s}_2' \mid \lambda(\mathsf{x}:(\gamma)_2(\sigma)).\,(\gamma)_2(\mathsf{e}_2))) \in \mathcal{V}\,[\![(\gamma)_R(\sigma) \multimap (\gamma)_R(\sigma')]\!]^j$$

Assume $j' \leq j$ and

$$((\mathsf{s}_1'' \mid \mathsf{v}_1), (\mathsf{s}_2'' \mid \mathsf{v}_2)) \in \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^{j'} \tag{21}$$

We now need to show

$$((\mathsf{s}_1 + \mathsf{s}_1' + \mathsf{s}_1'' \mid (\gamma)_1(\mathsf{e}_1)[\mathsf{v}_1/\mathsf{x}]), (\mathsf{s}_2 + \mathsf{s}_2' + \mathsf{s}_2'' \mid (\gamma)_2(\mathsf{e}_2)[\mathsf{v}_2/\mathsf{x}])) \in \mathcal{E}\,[\![(\gamma)_R(\sigma')]\!]^{j'}$$

To get this, we instantiate (19) with $((\mathsf{s}_1' + \mathsf{s}_1'', \mathsf{s}_1' + \mathsf{s}_1'') \mid \gamma[\mathsf{x} \mapsto (\mathsf{v}_1, \mathsf{v}_2)])$. It remains to show

$$((\mathsf{s}_1' + \mathsf{s}_1'', \mathsf{s}_1' + \mathsf{s}_1'') \mid \gamma[\mathsf{x} \mapsto (\mathsf{v}_1, \mathsf{v}_2)]) \in \mathcal{G}\,[\![\Gamma, \mathsf{x}{:}\sigma]\!]^{j'}$$

This follows from (20) and (21). □

Lemma 13 (Compat Unit).

$$!\Gamma \vdash_L (\emptyset \mid \langle\rangle) \lesssim^{log} (\emptyset \mid \langle\rangle) : 1 \qquad\qquad ((\emptyset \mid \langle\rangle), (\emptyset \mid \langle\rangle)) \in \mathcal{V}\,[\![1]\!]^j$$

Proof. The open case follows directly from the closed case, which is immediate from the definition of $\mathcal{V}\,[\![1]\!]^j$. □

Lemma 14 (Compat Unit Elimination).

$$\frac{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : 1 \qquad \Gamma' \vdash_L (\mathsf{s}_1' \mid \mathsf{e}_1') \lesssim^{log} (\mathsf{s}_2' \mid \mathsf{e}_2') : \sigma}{\Gamma \boxplus \Gamma' \vdash_L (\mathsf{s}_1 + \mathsf{s}_1' \mid \mathsf{e}_1; \mathsf{e}_1') \lesssim^{log} (\mathsf{s}_2 + \mathsf{s}_2' \mid \mathsf{e}_2; \mathsf{e}_2') : \sigma}$$

$$\frac{((\mathsf{s}_1 \mid \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{v}_2)) \in \mathcal{V}\,[\![1]\!]^j \qquad ((\mathsf{s}_1' \mid \mathsf{e}_1), (\mathsf{s}_2' \mid \mathsf{e}_2)) \in \mathcal{E}\,[\![\sigma]\!]^j}{((\mathsf{s}_1 + \mathsf{s}_1' \mid \mathsf{v}_1; \mathsf{e}_1), (\mathsf{s}_2 + \mathsf{s}_2' \mid \mathsf{v}_2; \mathsf{e}_2)) \in \mathcal{E}\,[\![\sigma]\!]^j}$$

Proof. The open case follows from the closed case using Lemma 7 (Splitting Lemma) and Lemma 8 (Monadic Bind). For the closed case, by inversion on the definition of $\mathcal{V}\,[\![1]\!]^j$ we get $\mathsf{v}_1 = \mathsf{v}_2 = \langle\rangle$ and $\mathsf{s}_1 = \mathsf{s}_2 = \emptyset$. Since $(\mathsf{s}_i' \mid \langle\rangle; \mathsf{e}_i) \overset{\mathsf{L}}{\hookrightarrow} (\mathsf{s}_i' \mid \mathsf{e}_i)$, by closure under anti-reduction it suffices to show $((\mathsf{s}_1' \mid \mathsf{e}_1), (\mathsf{s}_2' \mid \mathsf{e}_2)) \in \mathcal{E}\,[\![\sigma]\!]^j$, which we already know. □

Lemma 15 (Compat App).

$$\frac{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : \sigma' \multimap \sigma \qquad \Gamma' \vdash_L (\mathsf{s}_1' \mid \mathsf{e}_1') \lesssim^{log} (\mathsf{s}_2' \mid \mathsf{e}_2') : \sigma'}{\Gamma \boxplus \Gamma' \vdash_L (\mathsf{s}_1 + \mathsf{s}_1' \mid \mathsf{e}_1\ \mathsf{e}_1') \lesssim^{log} (\mathsf{s}_2 + \mathsf{s}_2' \mid \mathsf{e}_2\ \mathsf{e}_2') : \sigma}$$

Proof. Assume

$$\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma' \multimap \sigma \tag{22}$$

$$\Gamma' \vdash_L (s'_1 \mid e'_1) \lesssim^{log} (s'_2 \mid e'_2) : \sigma' \tag{23}$$

Consider $j \geq 0$, $((s_3, s_4) \mid \gamma) \in \mathcal{G}\,[\![\Gamma \boxplus \Gamma']\!]^j$. We need to show

$$((s_1 + s'_1 + s_3 \mid (\gamma)_1(e_1\ e'_1)), (s_2 + s'_2 + s_4 \mid (\gamma)_2(e_2\ e'_2))) \in \mathcal{E}\,[\![\gamma_R(\sigma)]\!]^j$$

From the Lemma 7 (Splitting Lemma) we get that there exists $s_5, s_6, \gamma', s'_5, s'_6, \gamma''$ such that

$$((s_5, s_6) \mid \gamma') \in \mathcal{G}\,[\![\Gamma]\!]^j \wedge ((s'_5, s'_6) \mid \gamma'') \in \mathcal{G}\,[\![\Gamma']\!]^j \tag{24}$$

$$s_3 = s_5 + s'_5 \wedge s_4 = s_6 + s'_6 \wedge \gamma = \gamma' \boxplus \gamma'' \tag{25}$$

Instantiating (22) and (23) with the left and right sides of (24) respectively we have

$$((s_1 + s_5 \mid (\gamma')_1(e_1)), (s_2 + s_6 \mid (\gamma')_2(e_2))) \in \mathcal{E}\,[\![\gamma'_R(\sigma') \multimap \gamma'_R(\sigma)]\!]^j$$

$$((s'_1 + s'_5 \mid (\gamma'')_1(e'_1)), (s'_2 + s'_6 \mid (\gamma'')_2(e'_2))) \in \mathcal{E}\,[\![\gamma''_R(\sigma')]\!]^j$$

Applying monadic bind twice, let $j' \leq j$ and

$$((s''_1 \mid v_1), (s''_2 \mid v_2)) \in \mathcal{V}\,[\![\gamma'_R(\sigma') \multimap \gamma'_R(\sigma)]\!]^{j'} \tag{26}$$

$$((s'''_1 \mid v'_1), (s'''_2 \mid v'_2)) \in \mathcal{V}\,[\![\gamma''_R(\sigma')]\!]^{j'} \tag{27}$$

It suffices to show

$$((s''_1 + s'''_1 \mid v_1\ v'_1), (s''_2 + s'''_2 \mid v_2\ v'_2)) \in \mathcal{E}\,[\![\gamma_R(\sigma)]\!]^{j'}$$

Note that $\gamma'_R(\sigma') = \gamma''_R(\sigma')$ and $\gamma_R(\sigma) = \gamma'_R(\sigma)$. We get what we need from instantiating (26) with (27). □

Lemma 16 (Compat Inj).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma_i}{\Gamma \vdash_L (s_1 \mid \mathsf{inj_i}\ e_1) \lesssim^{log} (s_2 \mid \mathsf{inj_i}\ e_2) : \sigma_1 \oplus \sigma_2} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}\,[\![\sigma_i]\!]^j}{((s_1 \mid \mathsf{inj_i}\ v_1), (s_2 \mid \mathsf{inj_i}\ v_2)) \in \mathcal{V}\,[\![\sigma_1 \oplus \sigma_2]\!]^j}$$

Proof. The open case follows from the closed case using Lemma 8 (Monadic Bind). The closed case is immediate from the definition. □

Lemma 17 (Compat Case).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma \oplus \sigma' \quad \Gamma', x{:}\sigma \vdash_L (s_3 \mid e_3) \lesssim^{log} (s_4 \mid e_4) : \sigma'' \quad \Gamma', x'{:}\sigma' \vdash_L (s_3 \mid e'_3) \lesssim^{log} (s_4 \mid e'_4) : \sigma''}{\Gamma \boxplus \Gamma' \vdash_L (s_1 + s_3 \mid \mathsf{case}\ e_1\ \mathsf{of}\ x.\ e_3 \mid x'.\ e'_3) \lesssim^{log} (s_2 + s_4 \mid \mathsf{case}\ e_2\ \mathsf{of}\ x.\ e_4 \mid x'.\ e'_4) : \sigma''}$$

Proof. Assume

$$\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma \oplus \sigma' \tag{28}$$

$$\Gamma', x{:}\sigma \vdash_L (s_3 \mid e_3) \lesssim^{log} (s_4 \mid e_4) : \sigma'' \tag{29}$$

$$\Gamma', x'{:}\sigma' \vdash_L (s_3 \mid e'_3) \lesssim^{log} (s_4 \mid e'_4) : \sigma'' \tag{30}$$

Consider $j \geq 0$, $((s_5'', s_6'') \mid \gamma'') \in \mathcal{G}\llbracket \Gamma \boxplus \Gamma' \rrbracket^j$. We need to show

$$((s_1 + s_3 + s_5'' \mid (\gamma'')_1(\text{case } e_1 \text{ of } x.\, e_3 \mid x'.\, e_3')), (s_2 + s_4 + s_6'' \mid (\gamma'')_2(\text{case } e_2 \text{ of } x.\, e_4 \mid x'.\, e_4'))) \in \mathcal{E}\llbracket \gamma''_R(\sigma) \rrbracket^j$$

From Lemma 7 (Splitting Lemma) we get that there exists $s_5, s_6, \gamma, s_5', s_6', \gamma'$ such that

$$((s_5, s_6) \mid \gamma) \in \mathcal{G}\llbracket \Gamma \rrbracket^j \tag{31}$$

$$((s_5', s_6') \mid \gamma') \in \mathcal{G}\llbracket \Gamma' \rrbracket^j \tag{32}$$

$$s_5'' = s_5 + s_5' \wedge s_6'' = s_6 + s_6' \wedge \gamma'' = \gamma \boxplus \gamma' \tag{33}$$

Instantiate (28) with (31). We have

$$((s_1 + s_5 \mid (\gamma)_1(e_1)), (s_2 + s_6 \mid (\gamma)_2(e_2))) \in \mathcal{E}\llbracket \gamma_R(\sigma) \oplus \gamma_R(\sigma') \rrbracket^j$$

Applying monadic bind, let $j' \leq j$ and

$$((s_1' \mid v_1), (s_2' \mid v_2)) \in \mathcal{V}\llbracket \gamma_R(\sigma) \oplus \gamma_R(\sigma') \rrbracket^{j'} \tag{34}$$

It suffices to show

$$((s_1' + s_3 + s_5' \mid \text{case } v_1 \text{ of } x.\, (\gamma')_1(e_3) \mid x'.\, (\gamma')_1(e_3')),$$
$$(s_2' + s_4 + s_6' \mid \text{case } v_2 \text{ of } x.\, (\gamma')_2(e_4) \mid x'.\, (\gamma')_2(e_4'))) \in \mathcal{E}\llbracket \gamma''_R(\sigma'') \rrbracket^j$$

Note that $\gamma'_R(\sigma') = \gamma''_R(\sigma')$ and $\gamma_R(\sigma) = \gamma'_R(\sigma)$.

**Case** $v_i = \text{inj}_1\, v_i'$

From (34) and the definition of $\mathcal{V}\llbracket \gamma_R(\sigma) \oplus \gamma_R(\sigma') \rrbracket^{j'}$,

$$((s_1' \mid v_1'), (s_2' \mid v_2')) \in \mathcal{V}\llbracket \gamma_R(\sigma') \rrbracket^{j'} \tag{35}$$

By closure under anti-reduction, it suffices to show

$$((s_1' + s_3 + s_5' \mid (\gamma')_1(e_3)[v_1/x]), (s_2' + s_4 + s_6' \mid (\gamma')_2(e_4)[v_2/x])) \in \mathcal{E}\llbracket \gamma''_R(\sigma'') \rrbracket^{j'}$$

(32) and (35) let us instantiate (29) with $\gamma'[x \mapsto (v_1', v_2')]$ to get this.

**Case** $v_i = \text{inj}_2\, v_i'$

Analagous to the previous case.

□

Lemma 18 (Compat Fold).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma[\mu\alpha.\, \sigma/\alpha]}{\Gamma \vdash_L (s_1 \mid \text{fold}_{\mu\alpha.\sigma}\, e_1) \lesssim^{log} (s_2 \mid \text{fold}_{\mu\alpha.\sigma}\, e_2) : \mu\alpha.\, \sigma} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}\llbracket \sigma[\mu\alpha.\, \sigma/\alpha] \rrbracket^{j-1}}{((s_1 \mid \text{fold}_{\mu\alpha.\sigma}\, v_1), (s_2 \mid \text{fold}_{\mu\alpha.\sigma}\, v_2)) \in \mathcal{V}\llbracket \mu\alpha.\, \sigma \rrbracket^j}$$

Proof. The open case follows from the closed case using Lemma 8 (Monadic Bind). The closed case follows from from the definition of $\mathcal{V}\llbracket \mu\alpha.\, \sigma \rrbracket^j$ and Lemma 3 (Monotonicity). □

Lemma 19 (Compat Unfold).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \mu\alpha.\,\sigma}{\Gamma \vdash_L (s_1 \mid \mathsf{unfold}\, e_1) \lesssim^{log} (s_2 \mid \mathsf{unfold}\, e_2) : \sigma[\mu\alpha.\,\sigma/\alpha]} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}\,[\![\mu\alpha.\,\sigma]\!]^j}{((s_1 \mid \mathsf{unfold}\, v_1), (s_2 \mid \mathsf{unfold}\, v_2)) \in \mathcal{E}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^j}$$

Proof. The open case follows from the closed case using Lemma 8 (Monadic Bind). For the closed case, by inversion on the definition of $\mathcal{V}\,[\![\mu\alpha.\,\sigma]\!]^j$ we have $v_i = \mathsf{fold}_{\mu\alpha.\sigma}\, v'_i$ and

$$\forall j' < j.((s_1 \mid v'_1), (s_2 \mid v'_2)) \in \mathcal{V}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j'} \tag{36}$$

Since $(s_i \mid \mathsf{unfold}\, \mathsf{fold}_{\mu\alpha.\sigma}\, v'_i) \overset{L}{\hookrightarrow}{}^1 (s_i \mid v'_i)$, by closure under anti-reduction it suffices to show

$$((s_1 \mid v'_1), (s_2 \mid v'_2)) \in \mathcal{E}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j-1}$$

From Lemma 1, $\mathcal{E}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j-1} \supseteq \mathcal{V}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j-1}$. Therefore we need only show $((s_1 \mid v'_1), (s_2 \mid v'_2)) \in \mathcal{V}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]^{j-1}$ which follows from (36). □

Lemma 20 (Compat Share).

$$\frac{!\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma}{!\Gamma \vdash_L (\emptyset \mid \mathsf{share}(s_1 : \Psi).\, e_1) \lesssim^{log} (\emptyset \mid \mathsf{share}(s_2 : \Psi).\, e_2) : !\sigma}$$

Proof. Assume

$$!\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma \tag{37}$$

Consider $j \geq 0$ and $((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma]\!]^j$. We need to show

$$((\emptyset \mid \mathsf{share}(s_1 : (\gamma)_1(\Psi)).\, (\gamma)_1(e_1)), (\emptyset \mid \mathsf{share}(s_2 : (\gamma)_2(\Psi)).\, (\gamma)_2(e_2))) \in \mathcal{E}\,[\![!(\gamma)_R(\sigma)]\!]^j$$

Consider $j' \leq j$, $(s''_1 \mid v''_1)$ such that $(\emptyset \mid \mathsf{share}(s_1 : (\gamma)_1(\Psi)).\, (\gamma)_1(e_1)) \overset{L}{\hookrightarrow}{}^{j'} (s''_1 \mid v''_1)$. It suffices to show

$$\exists v''_2.(\emptyset \mid \mathsf{share}(s_2 : (\gamma)_2(\Psi)).\, (\gamma)_2(e_2)) \overset{L}{\hookrightarrow}{}^* (\emptyset \mid v''_2) \wedge ((s_1 \mid v''_1), (\emptyset \mid v''_2)) \in \mathcal{V}\,[\![!(\gamma)_R(\sigma)]\!]^{j-j'}$$

Note that a share expression can only reduce to another share expression, which must be paired with the empty store. Therefore, $s''_1 = \emptyset$ and

$$\exists s'_1, \Psi''_1, v'_1.v''_1 = \mathsf{share}(s'_1 : \Psi''_1).\, v'_1 \wedge (s_1 \mid (\gamma)_1(e_1)) \overset{L}{\hookrightarrow}{}^{j'} (s'_1 \mid v'_1)$$

From (37), we have that there exists $(s'_2 \mid v'_2)$ such that $(s_2 \mid (\gamma)_1(e_2)) \overset{L}{\hookrightarrow}{}^* (s'_2 \mid v'_2)$ and

$$((s'_1 \mid v'_1), (s'_2 \mid v'_2)) \in \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^{j-j'} \tag{38}$$

Since only well-typed terms can be related, there exists $\Psi''_2$ such that $\Psi''_2; \cdot \vdash_L s'_2 \mid v'_2 : (\gamma)_2(\sigma)$. Note that $(\emptyset \mid \mathsf{share}(s_2 : (\gamma)_2(\Psi)).\, (\gamma)_2(e_2)) \overset{L}{\hookrightarrow}{}^* (\emptyset \mid \mathsf{share}(s'_2 : \Psi''_2).\, v'_2)$. By closure under anti-reduction, it suffices to show

$$((\emptyset \mid \mathsf{share}(s'_1 : \Psi''_1).\, v'_1), (\emptyset \mid \mathsf{share}(s'_2 : \Psi''_2).\, v'_2)) \in \mathcal{E}\,[\![!(\gamma)_R(\sigma)]\!]^{j-j'} \supseteq \mathcal{V}\,[\![!(\gamma)_R(\sigma)]\!]^{j-j'}$$

But this follows from (38). □

Lemma 21 (Compat Copy).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : !\sigma}{\Gamma \vdash_L (s_1 \mid \mathsf{copy}^\sigma\ e_1) \lesssim^{log} (s_2 \mid \mathsf{copy}^\sigma\ e_2) : \sigma}$$

Proof. Assume

$$\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : !\sigma \tag{39}$$

Consider $j \geq 0$, $((s_1', s_2') \mid \gamma) \in \mathcal{G}\,[\![\Gamma]\!]^j$. We need to show

$$((s_1 + s_1' \mid \mathsf{copy}^\sigma\ (\gamma)_1(e_1)), (s_2 + s_2' \mid \mathsf{copy}^\sigma\ (\gamma)_2(e_2))) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^j$$

Instantiating (39), we get

$$((s_1 + s_1' \mid (\gamma)_1(e_1)), (s_2 + s_2' \mid (\gamma)_2(e_2))) \in \mathcal{E}\,[\![!\sigma]\!]^j \tag{40}$$

Assume $j' \leq j$ and $((\emptyset \mid v_1), (\emptyset \mid v_2)) \in \mathcal{V}\,[\![!\sigma]\!]^{j'}$. By monadic bind, it suffices to show

$$((\emptyset \mid \mathsf{copy}^\sigma\ v_1), (\emptyset \mid \mathsf{copy}^\sigma\ v_2)) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^{j'}$$

From the definition of $\mathcal{V}\,[\![!\sigma]\!]^{j'}$, we have that $v_i = \mathsf{share}(s_i'' : \Psi_i').\, v_i'$ where $((s_1'' \mid v_1'), (s_2'' \mid v_2')) \in \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^{j'}$. Therefore we need only show

$$((\emptyset \mid \mathsf{copy}^\sigma\ \mathsf{share}(s_1'' : \Psi_1').\, v_1'), (\emptyset \mid \mathsf{copy}^\sigma\ \mathsf{share}(s_2'' : \Psi_2').\, v_2')) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^{j'}$$

This follows from Lemma 10 (Copy). ◻

Lemma 22 (Compat Location Dead).

$$!\Gamma \vdash_L ([\ell \mapsto \cdot] \mid \ell) \lesssim^{log} ([\ell \mapsto \cdot] \mid \ell) : \mathsf{Box}\ 0$$

Proof. Consider $j \geq 0$ and $((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma]\!]^j$. We need to show

$$(([\ell \mapsto \cdot] \mid \ell), ([\ell \mapsto \cdot] \mid \ell)) \in \mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j \supseteq \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$$

But $(([\ell \mapsto \cdot] \mid \ell), ([\ell \mapsto \cdot] \mid \ell)) \in \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$ is immediate. ◻

Lemma 23 (Compat Location Live).

$$\frac{\cdot \vdash_L (s_1 \mid v_1) \lesssim^{log} (s_2 \mid v_2) : \sigma}{!\Gamma \vdash_L ([\ell \mapsto (s_1 \mid v_1)] \mid \ell) \lesssim^{log} ([\ell \mapsto (s_2 \mid v_2)] \mid \ell) : \mathsf{Box}\ 1\ \sigma}$$

Proof. Assume

$$\cdot \vdash_L (s_1 \mid v_1) \lesssim^{log} (s_2 \mid v_2) : \sigma \tag{41}$$

Consider $j \geq 0$ and $((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma]\!]^j$. We need to show

$$(([\ell \mapsto (s_1 \mid v_1)] \mid \ell), ([\ell \mapsto (s_2 \mid v_2)] \mid \ell)) \in \mathcal{E}\,[\![\mathsf{Box}\ 1\ (\gamma)_R(\sigma)]\!]^j \supseteq \mathcal{V}\,[\![\mathsf{Box}\ 1\ (\gamma)_R(\sigma)]\!]^j$$

It suffices to show $((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^j$. By Lemma 1, we need only prove $((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{E}\,[\![(\gamma)_R(\sigma)]\!]^j$, which follows from (41). ◻

Lemma 24 (Compat Free).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \mathsf{Box}\ 0}{\Gamma \vdash_L (s_1 \mid \mathsf{free}\ e_1) \lesssim^{log} (s_2 \mid \mathsf{free}\ e_2) : 1} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j}{((s_1 \mid \mathsf{free}\ v_1), (s_2 \mid \mathsf{free}\ v_2)) \in \mathcal{E}\,[\![1]\!]^j}$$

Proof. By Lemma 8 (Monadic Bind), the open case reduces to the closed case. We need to show $((\mathsf{s}_1 \mid \mathsf{free}\ \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{free}\ \mathsf{v}_2)) \in \mathcal{E}\,[\![1]\!]^j$. By inversion on the definition of $\mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$, $\mathsf{s}_i = [\ell_i \mapsto \cdot] \land \mathsf{v}_i = \ell_i$. Note that $([\ell_i \mapsto \cdot] \mid \mathsf{free}\ \ell_i) \overset{\mathsf{L}}{\hookrightarrow} (\emptyset \mid \langle\rangle)$ so by closure under anti-reduction it suffices to show

$$((\emptyset \mid \langle\rangle), (\emptyset \mid \langle\rangle)) \in \mathcal{E}\,[\![1]\!]^j \supseteq \mathcal{V}\,[\![1]\!]^j$$

Which is immediate. ◻

Lemma 25 (Compat New).

$$\frac{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : 1}{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{new}\ \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{new}\ \mathsf{e}_2) : \mathsf{Box}\ 0} \qquad \frac{((\mathsf{s}_1 \mid \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{v}_2)) \in \mathcal{V}\,[\![1]\!]^j}{((\mathsf{s}_1 \mid \mathsf{new}\ \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{new}\ \mathsf{v}_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j}$$

Proof. By Lemma 8 (Monadic Bind), the open case reduces to the closed case. By inversion on the definition of $\mathcal{V}\,[\![1]\!]^j$, $\mathsf{s}_1 = \mathsf{s}_2 = \emptyset$ and $\mathsf{v}_1 = \mathsf{v}_2 = \langle\rangle$. We need to show

$$((\emptyset \mid \mathsf{new}\ \langle\rangle), (\emptyset \mid \mathsf{new}\ \langle\rangle)) \in \mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j$$

By closure under anti-reduction, it is sufficient to show

$$(([\ell_1 \mapsto \cdot] \mid \ell_1), ([\ell_2 \mapsto \cdot] \mid \ell_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 0]\!]^j \supseteq \mathcal{V}\,[\![\mathsf{Box}\ 0]\!]^j$$

Which is immediate. ◻

Lemma 26 (Compat Box).

$$\frac{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : (\mathsf{Box}\ 0) \otimes \sigma}{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{box}\ \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{box}\ \mathsf{e}_2) : \mathsf{Box}\ 1\ \sigma} \qquad \frac{((\mathsf{s}_1 \mid \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{v}_2)) \in \mathcal{V}\,[\![(\mathsf{Box}\ 0) \otimes \rho]\!]^j}{((\mathsf{s}_1 \mid \mathsf{box}\ \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{box}\ \mathsf{v}_2)) \in \mathcal{E}\,[\![\mathsf{Box}\ 1\ \rho]\!]^j}$$

Proof. By Lemma 8 (Monadic Bind), the open case reduces to the closed case.

For the closed case, by inversion on the definition of $\mathcal{V}\,[\![(\mathsf{Box}\ 0) \otimes \rho]\!]^j$, we know $\mathsf{s}_i = \mathsf{s}'_i[\ell_i \mapsto \cdot]$ and $\mathsf{v}_i = \langle \ell_i, \mathsf{v}'_i\rangle$, with $((\mathsf{s}'_1 \mid \mathsf{v}'_1), (\mathsf{s}'_2 \mid \mathsf{v}'_2)) \in \mathcal{V}\,[\![\rho]\!]^j$. Inspecting the operational semantics, we see

$$(\mathsf{s}'_i[\ell_i \mapsto \cdot] \mid \mathsf{box}\ \langle \ell_i, \mathsf{v}'_i\rangle) \overset{\mathsf{L}}{\rightsquigarrow} ([\ell_i \mapsto (\mathsf{s}'_i \mid \mathsf{v}'_i)] \mid \ell_i)$$

So it is sufficient to show

$$(([\ell_1 \mapsto (\mathsf{s}'_1 \mid \mathsf{v}'_1)] \mid \ell_1), ([\ell_2 \mapsto (\mathsf{s}'_2 \mid \mathsf{v}'_2)] \mid \ell_2)) \in \mathcal{V}\,[\![\mathsf{Box}\ 1\ \rho]\!]^{j-1}$$

Which holds immediately by assumption and Lemma 3 (Monotonicity). ◻

Lemma 27 (Compat Unbox).

$$\frac{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{e}_2) : \mathsf{Box}\ 1\ \sigma}{\Gamma \vdash_L (\mathsf{s}_1 \mid \mathsf{unbox}\ \mathsf{e}_1) \lesssim^{log} (\mathsf{s}_2 \mid \mathsf{unbox}\ \mathsf{e}_2) : (\mathsf{Box}\ 0) \otimes \sigma} \qquad \frac{((\mathsf{s}_1 \mid \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{v}_2)) \in \mathcal{V}\,[\![\mathsf{Box}\ 1\ \rho]\!]^j}{((\mathsf{s}_1 \mid \mathsf{unbox}\ \mathsf{v}_1), (\mathsf{s}_2 \mid \mathsf{unbox}\ \mathsf{v}_2)) \in \mathcal{E}\,[\![(\mathsf{Box}\ 0) \otimes \rho]\!]^j}$$

Proof. By Lemma 8 (Monadic Bind), the open case reduces to the closed case.

By inversion on $\mathcal{V}\,[\![\mathsf{Box}\ 1\ \rho]\!]^j$, we know $\mathsf{s}_i = [\ell_i \mapsto (\mathsf{s}'_i \mid \mathsf{v}'_i)]$ and $\mathsf{v}_i = \ell_i$, with $((\mathsf{s}'_1 \mid \mathsf{v}'_1), (\mathsf{s}'_2 \mid \mathsf{v}'_2)) \in \mathcal{V}\,[\![\rho]\!]^j$.

Then inspecting the operational semantics we see

$$([\ell_i \mapsto (\mathsf{s}'_i \mid \mathsf{v}'_i)] \mid \mathsf{unbox}\ \ell_i) \overset{\mathsf{L}}{\rightsquigarrow} (\mathsf{s}'_i[\ell_i \mapsto \cdot] \mid \langle \ell_i, \mathsf{v}'_i\rangle)$$

So it is sufficient to show

$$((s_1'[\ell_1 \mapsto \cdot] \mid \langle \ell_1, v_1' \rangle), (s_2'[\ell_2 \mapsto \cdot] \mid \langle \ell_2, v_2' \rangle)) \in \mathcal{V} [\![(\mathsf{Box}\ 0) \otimes \rho]\!]^{j-1}$$

Which holds immediately by assumption and Lemma 3 (Monotonicity). ◻

Lemma 28 (Compat LU Boundary).

$$\frac{!\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma}{!\Gamma \vdash_L (\emptyset \mid \mathcal{LU}(e_1)) \lesssim^{log} (\emptyset \mid \mathcal{LU}(e_1)) : ![\sigma]} \qquad \frac{(v_1, v_2) \in \mathcal{V} [\![\rho]\!]^j}{((\emptyset \mid \mathcal{LU}(v_1)), (\emptyset \mid \mathcal{LU}(v_2))) \in \mathcal{E} [\![![\rho]]\!]^j}$$

Proof. By instantiating and Lemma 8 (Monadic Bind), the closed case implies the open.
By the operational semantics, we have

$$\mathcal{LU}(v_i) \overset{L}{\rightsquigarrow} \mathsf{share}(\emptyset : \cdot).\ [v_i]$$

So it is sufficient to show

$$((\emptyset \mid \mathsf{share}(\emptyset : \cdot).\ [v_1]), (\emptyset \mid \mathsf{share}(\emptyset : \cdot).\ [v_2])) \in \mathcal{V} [\![![\rho]]\!]^{j-1}$$

Which follows by assumption, definition of the relation and Lemma 3 (Monotonicity). ◻

Lemma 29 (Compat UL Boundary).

$$\frac{!\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : ![\sigma]}{!\Gamma \vdash \mathcal{UL}(s_1 : \Psi_1 \mid e_1) \lesssim^{log} \mathcal{UL}(s_2 : \Psi_2 \mid e_2) : \sigma} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V} [\![![\rho]]\!]^j}{(\mathcal{UL}(s_1 : \Psi_1 \mid v_1), \mathcal{UL}(s_2 : \Psi_2 \mid v_2)) \in \mathcal{E} [\![\rho]\!]^j}$$

Proof. By instantiating quantifiers and Lemma 8 (Monadic Bind), the closed case implies the open case.
By definition of $\mathcal{V} [\![![\sigma]]\!]^j$, we know $s_1 = s_2 = \emptyset$, and $v_i = \mathsf{share}(\emptyset : \cdot).\ [v_i]$, where

$$(v_1, v_2) \in \mathcal{V} [\![\rho]\!]^j \tag{42}$$

and by the operational semantics, we have

$$\mathcal{UL}(\emptyset : \cdot \mid \mathsf{share}(\emptyset : \cdot).\ [v_i]) \overset{U}{\hookrightarrow} v_i$$

So it is sufficient to show $(v_1, v_2) \in \mathcal{V} [\![\rho]\!]^{j-1}$, and the result holds by Lemma 3 (Monotonicity) and (42). ◻

Lemma 30 (Compat Lump).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma \qquad \cdot \vdash_{UL} \sigma \simeq \sigma}{\Gamma \vdash_L (s_1 \mid \mathsf{lump}^\sigma\ e_1) \lesssim^{log} (s_2 \mid \mathsf{lump}^\sigma\ e_2) : ![\sigma]}$$

Proof. Follows by Lemma 8 (Monadic Bind) and Lemma 32 (Lumping/Unlumping Lemma). ◻

Lemma 31 (Compat Unlump).

$$\frac{\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : ![\sigma] \qquad \cdot \vdash_{UL} \sigma \simeq \sigma}{\Gamma \vdash_L (s_1 \mid {}^\sigma\mathsf{unlump}\ e_1) \lesssim^{log} (s_2 \mid {}^\sigma\mathsf{unlump}\ e_2) : \sigma}$$

Proof. Follows by Lemma 8 (Monadic Bind) and Lemma 32 (Lumping/Unlumping Lemma). ◻

Lemma 32 (Lumping/Unlumping Lemma).

$$\frac{((\emptyset \mid v_1),(\emptyset \mid v_2)) \in \mathcal{V}[\![\rho]\!]^j \qquad v_1 \leftarrow^{(\rho)_1} v_1 \qquad v_2 \leftarrow^{(\rho)_2} v_2 \qquad \cdot \vdash_{UL} \rho \simeq \rho}{(v_1,v_2) \in \mathcal{V}[\![\rho]\!]^j}$$

$$\frac{(v_1,v_2) \in \mathcal{V}[\![\rho]\!]^j \qquad v_1 \rightarrow^{(\rho)_1} v_1 \qquad v_2 \rightarrow^{(\rho)_2} v_2 \qquad \cdot \vdash_{UL} \rho \simeq \rho}{((\emptyset \mid v_1),(\emptyset \mid v_2)) \in \mathcal{V}[\![\rho]\!]^j}$$

Proof. We prove the two statements by mutual induction, by parallel induction on the derivations of $v_1 \leftrightarrow {}^{\rho} v_1, v_2 \leftrightarrow {}^{\rho} v_2$.

**Case** $v_i = \langle\rangle \leftrightarrow {}^{!1}\mathsf{share}\,\langle\rangle = v_i$: Immediate from definitions.

**Case** $v_i = \langle v_i', v_i'' \rangle \leftrightarrow {}^{!(\sigma_i' \otimes \sigma_i'')}\mathsf{share}(s_i' + s_i'' : \Psi_i' \uplus \Psi_i'').\, \langle v_i', v_i'' \rangle = v_i$: Immediate.

**Case** $v_i = \mathsf{inj_k}\, v \leftrightarrow {}^{!(\sigma_1 \oplus \sigma_2)}\mathsf{share}(s : \Psi).\, \mathsf{inj_k}\, v = v_i$: Immediate.

**Case** $v_i \rightarrow {}^{!(!\sigma \multimap !\sigma')}\mathsf{share}(\emptyset : \cdot).\, \lambda(x : !\sigma).\, {}^{\sigma'}\mathcal{LU}(v_i\, \mathcal{UL}(\emptyset{:}\cdot \mid \mathsf{lump}^{\sigma}\, x)) = v_i$:

By definition of the relation, it is sufficient to show that for any $((\emptyset \mid v_1'), (\emptyset \mid v_2')) \in \mathcal{V}[\![!\sigma]\!]^j$,

$$((\emptyset \mid {}^{\sigma'}\mathcal{LU}(v_1\, \mathcal{UL}(\emptyset{:}\cdot \mid \mathsf{lump}^{\sigma}\, v_1'))), (\emptyset \mid {}^{\sigma'}\mathcal{LU}(v_2\, \mathcal{UL}(\emptyset{:}\cdot \mid \mathsf{lump}^{\sigma}\, v_2')))) \in \mathcal{E}[\![!\sigma']\!]^j$$

The result then follows from the inductive hypothesis for $\sigma, \sigma'$ and compatibility lemmas Lemma 15 (Compat App), Lemma 29 (Compat UL Boundary), Lemma 28 (Compat LU Boundary), using Lemma 8 (Monadic Bind) where needed.

**Case** $v_i = \lambda(x : \sigma).\, \mathcal{UL}(\emptyset{:}\cdot \mid \mathsf{lump}^{\sigma'}\, (v_i\, ({}^{\sigma}\mathsf{unlump}\, \mathcal{LU}(x)))) \leftarrow {}^{!(\sigma \multimap \sigma')} v_i$: Symmetric argument to previous case

**Case** $v_i \leftrightarrow {}^{[\sigma]}\mathsf{share}(\emptyset : \cdot).\, [v_i] = v_i$ Immediate.

**Case** $v_i \leftrightarrow {}^{!!\sigma}\mathsf{share}(\emptyset : \cdot).\, \mathsf{share}(\emptyset : \cdot).\, v_i' = v_i$ Immediate.

**Case** $v_i \leftrightarrow {}^{!\mathsf{Box}\ 1\ \sigma}\mathsf{share}([\ell_i \mapsto (s_i \mid v_i')] : (\cdot; \ell_i \vdash \Psi_i : \mathsf{Box}\ 1\ \sigma)).\, \ell_i = v_i$ Immediate.

**Case** $v_i = \mathsf{fold}_{\mu\alpha.\sigma}\, v_i' \leftrightarrow {}^{!\mu\alpha.\sigma}\mathsf{share}(s_i : \Psi_i).\, (\mathsf{fold}_{\mu\alpha.\sigma}\, v_i') = v_i$ Immediate.

□

Lemma 33 (Compat Tensor).

$$\frac{\Gamma' \vdash_L (s_1' \mid e_1') \lesssim^{log} (s_2' \mid e_2') : \sigma' \qquad \Gamma'' \vdash_L (s_1'' \mid e_1'') \lesssim^{log} (s_2'' \mid e_2'') : \sigma''}{\Gamma' \boxplus \Gamma'' \vdash_L (s_1' + s_1'' \mid \langle e_1', e_1'' \rangle) \lesssim^{log} (s_2' + s_2'' \mid \langle e_2', e_2'' \rangle) : \sigma' \otimes \sigma''}$$

$$\frac{((s_1' \mid v_1'),(s_2' \mid v_2')) \in \mathcal{V}[\![\sigma']\!]^j \qquad ((s_1'' \mid v_1''),(s_2'' \mid v_2'')) \in \mathcal{V}[\![\sigma'']\!]^j}{((s_1' + s_1'' \mid \langle v_1', v_1'' \rangle),(s_2' + s_2'' \mid \langle v_2', v_2'' \rangle)) \in \mathcal{V}[\![\sigma' \otimes \sigma'']\!]^j}$$

Proof. The open case follows from the closed case using Lemma 7 (Splitting Lemma) Lemma 8 (Monadic Bind) twice. The closed case is immediate from the definition. □

Lemma 34 (Compat Tensor Elimination).

$$\frac{\Gamma_l \vdash_L (s_{l,s,1} \mid e_{l,1}) \lesssim^{log} (s_{l,s,2} \mid e_{l,2}) : \sigma_a \otimes \sigma_b \qquad \Gamma_r, x_a{:}\sigma_a, x_b{:}\sigma_b \vdash_L (s_{r,s,1} \mid e_{r,1}) \lesssim^{log} (s_{r,s,2} \mid e_{r,2}) : \sigma}{\Gamma_l \boxplus \Gamma_r \vdash_L (s_{l,s,1} + s_{r,s,1} \mid \mathsf{let}\,\langle x_a, x_b \rangle = e_{l,1}\,\mathsf{in}\,e_{r,1}) \lesssim^{log} (s_{l,s,2} + s_{r,s,2} \mid \mathsf{let}\,\langle x_a, x_b \rangle = e_{l,2}\,\mathsf{in}\,e_{r,2}) : \sigma}$$

Proof. Naming convention is as follows: $l, r$ indicates if it is in the left subterm (discriminee) or right subterm (continuation); $d, s$ indicates if it is a dynamic store or static store; $1, 2$ indicates if it is on the less than or greater than side of the approximation judgment, $a, b$ indicates if it is in the $a$ or $b$ side of the tensor $\sigma_a \otimes \sigma_b$.

Assume $((s_{d,1}, s_{d,2}) \mid \gamma) \in \mathcal{G}[\![\Gamma_l \boxplus \Gamma_r]\!]^j$. By Lemma 7 (Splitting Lemma), we have $s_{d,i} = s_{l,d,i} + s_{r,d,i}$ $\gamma = \gamma_l \boxplus \gamma_r$ with $((s_{l,d,1}, s_{l,d,2}) \mid \gamma_l) \in \mathcal{G}[\![\Gamma_l]\!]^j$, $((s_{r,d,1}, s_{r,d,2}) \mid \gamma_l) \in \mathcal{G}[\![\Gamma_r]\!]^j$.

By inductive hypothesis and Lemma 6 (Splitting and Relational Substitution), we have

$$((s_{l,s,1} + s_{l,d,1} \mid (\gamma_l)_1(e_{l,1})), (s_{l,s,2} + s_{l,d,2} \mid (\gamma_l)_1(e_{l,2}))) \in \mathcal{E}[\![(\gamma_l)_R(\sigma_a \otimes \sigma_b)]\!]^j$$

And we seek to prove that

$$((s_{l,s,1} + s_{r,s,1} + s_{l,d,1} + s_{r,d,1} \mid \mathsf{let}\,\langle x_a, x_b\rangle = (\gamma_l)_1(e_{l,1})\,\mathsf{in}\,(\gamma_r)_1(e_{r,1})), (s_{l,s,2} + s_{r,s,2} + s_{l,d,2} + s_{r,d,2} \mid \mathsf{let}\,\langle x_a, x_b\rangle = (\gamma_l)_2(e_{l,2})\,\mathsf{in}\,(\gamma_r)_2(e_{r,2}))) \in \mathcal{E}[\![(\gamma)_R(\sigma)]\!]^j$$

By Lemma 8 (Monadic Bind) and definition of $\mathcal{V}[\![- \otimes -]\!]^-$, it is sufficient to prove that for some $j' \leq j$, $((s_{l,d,1,a} \mid v_{l,1,a}), (s_{l,d,2,a} \mid v_{l,2,a})) \in \mathcal{V}[\![\sigma_a]\!]^{j'}$, $((s_{l,d,1,b} \mid v_{l,1,b}), (s_{l,d,2,b} \mid v_{l,2,b})) \in \mathcal{V}[\![\sigma_b]\!]^{j'}$,

$$((s_{r,s,1} + s_{l,d,1,a} + s_{l,d,1,b} + s_{r,d,1} \mid \mathsf{let}\,\langle x_a, x_b\rangle = \langle v_{l,1,a}, v_{l,1,b}\rangle\,\mathsf{in}\,(\gamma_r)_1(e_{r,1})), (s_{r,s,2} + s_{l,d,1,a} + s_{l,d,1,b} + s_{r,d,2} \mid \mathsf{let}\,\langle x_a, x_b\rangle = \langle v_{l,1,a}, v_{l,1,b}\rangle\,\mathsf{in}\,(\gamma_r)_2(e_{r,2}))) \in \mathcal{E}[\![(\gamma)_R(\sigma)]\!]^{j'}$$

By the operational semantics,

$$(s_{r,s,1} + s_{l,d,1,a} + s_{l,d,1,b} + s_{r,d,1} \mid \mathsf{let}\,\langle x_a, x_b\rangle = \langle v_{l,1,a}, v_{l,1,b}\rangle\,\mathsf{in}\,(\gamma_r)_1(e_{r,1})) \overset{\mathsf{L}}{\rightsquigarrow} (s_{r,s,1} + s_{l,d,1,a} + s_{l,d,1,b} + s_{r,d,1} \mid (\gamma')_1(e_{r,1}))$$

$$(s_{r,s,2} + s_{l,d,2,a} + s_{l,d,2,b} + s_{r,d,2} \mid \mathsf{let}\,\langle x_a, x_b\rangle = \langle v_{l,2,a}, v_{l,2,b}\rangle\,\mathsf{in}\,(\gamma_r)_2(e_{r,2})) \overset{\mathsf{L}}{\rightsquigarrow} (s_{r,s,2} + s_{l,d,2,a} + s_{l,d,2,b} + s_{r,d,2} \mid (\gamma')_2(e_{r,2}))$$

where we define $\gamma' = \gamma[x_a \mapsto (v_{l,1,a}, v_{l,2,a})][x_b \mapsto (v_{l,1,b}, v_{l,2,b})]$

So we need to show

$$((s_{r,s,1} + s_{l,d,1,a} + s_{l,d,1,b} + s_{r,d,1} \mid (\gamma')_1(e_{r,1})), (s_{r,s,2} + s_{l,d,2,a} + s_{l,d,2,b} + s_{r,d,2} \mid (\gamma')_2(e_{r,2}))) \in \mathcal{E}[\![(\gamma)_R(\sigma)]\!]^{j'} = \mathcal{E}[\![(\gamma')_R(\sigma)]\!]^{j'}$$

So the result hold by inductive hypothesis and using Lemma 3 (Monotonicity), the fact that

$$((s_{l,d,1,a} + s_{l,d,1,b} + s_{r,d,1}, s_{l,d,2,a} + s_{l,d,2,b} + s_{r,d,2}) \mid \gamma') \in \mathcal{G}[\![\Gamma_r, x_a : \sigma_a, x_b : \sigma_b]\!]^{j'}$$

□

Lemma 35 (U Compatibility).

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash x \lesssim^{log} x : \sigma}$$

$$\frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma \qquad \Gamma \vdash e_1' \lesssim^{log} e_1' : \sigma'}{\Gamma \vdash \langle e_1, e_1' \rangle \lesssim^{log} \langle e_2, e_2' \rangle : \sigma \times \sigma'} \qquad \frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma_1 \times \sigma_2}{\Gamma \vdash \pi_i\, e_1 \lesssim^{log} \pi_i\, e_2 : \sigma_i}$$

$$\Gamma \vdash \langle\rangle \lesssim^{log} \langle\rangle : 1 \qquad \frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : 1 \qquad \Gamma \vdash e_1' \lesssim^{log} e_2' : \sigma}{\Gamma \vdash e_1; e_1' \lesssim^{log} e_2; e_2' : \sigma}$$

$$\frac{\Gamma, x{:}\sigma \vdash e_1 \lesssim^{log} e_2 : \sigma'}{\Gamma \vdash \lambda(x{:}\sigma).\, e_1 \lesssim^{log} \lambda(x{:}\sigma).\, e_2 : \sigma \to \sigma'} \qquad \frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma' \to \sigma \qquad \Gamma \vdash e_1' \lesssim^{log} e_2' : \sigma'}{\Gamma \vdash e_1'\, e_1' \lesssim^{log} e_2'\, e_2' : \sigma}$$

$$\frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma_i}{\Gamma \vdash \mathsf{inj_i}\, e_1 \lesssim^{log} \mathsf{inj_i}\, e_2 : \sigma_1 + \sigma_2} \qquad \frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma + \sigma' \qquad \begin{array}{c}\Gamma, x : \sigma \vdash e_3 \lesssim^{log} e_4 : \sigma'' \\ \Gamma, x' : \sigma' \vdash e_3' \lesssim^{log} e_4' : \sigma''\end{array}}{\Gamma \vdash \mathsf{case}\, e_1\, \mathsf{of}\, x.\, e_3 \mid x'.\, e_3' \lesssim^{log} \mathsf{case}\, e_2\, \mathsf{of}\, x.\, e_4 \mid x'.\, e_4' : \sigma''}$$

$$\frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma[\mu\alpha.\, \sigma/\alpha]}{\Gamma \vdash \mathsf{fold}_{\mu\alpha.\sigma}\, e_1 \lesssim^{log} \mathsf{fold}_{\mu\alpha.\sigma}\, e_2 : \mu\alpha.\, \sigma} \qquad \frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \mu\alpha.\, \sigma}{\Gamma \vdash \mathsf{unfold}\, e_1 \lesssim^{log} \mathsf{unfold}\, e_2 : \sigma[\mu\alpha.\, \sigma/\alpha]}$$

$$\frac{\Gamma, \alpha \vdash v_1 \lesssim^{log} v_2 : \sigma}{\Gamma \vdash \Lambda\alpha.\, v_1 \lesssim^{log} \Lambda\alpha.\, v_2 : \forall\alpha.\, \sigma} \qquad \frac{\Gamma \vdash e_1 \lesssim^{log} e_2 : \forall\alpha.\, \sigma \qquad \Gamma \vdash \sigma'}{\Gamma \vdash e_1\, [\sigma'] \lesssim^{log} e_2\, [\sigma'] : \sigma[\sigma'/\alpha]}$$

Proof. Two cases are proven below. The rest of the proofs of these properties are standard, and similar to those for L. □

Lemma 36 (Compat Type Abstraction).

$$\frac{!\Gamma, \alpha \vdash_v v_1 \lesssim^{log} v_2 : \forall\alpha.\, \sigma}{!\Gamma \vdash_v \Lambda\alpha.\, v_1 \lesssim^{log} \Lambda\alpha.\, v_2 : \forall\alpha.\, \sigma}$$

Proof. Given $((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\, [\![!\Gamma]\!]^j$, it is sufficient to show

$$(\Lambda\alpha.\, (\gamma)_1(v_1), \Lambda\alpha.\, (\gamma)_1(v_2)) \in \mathcal{V}\, [\![\forall\alpha.\, (\gamma)_R(\sigma)]\!]^j$$

which is equivalent to showing for any $\sigma_1, \sigma_2, R \in \mathrm{Rel}[\sigma_1, \sigma_2]$ that

$$((\gamma')_1(v_1), (\gamma')_1(v_2)) \in \mathcal{V}\, [\![(\gamma')_R(\sigma)]\!]^j$$

where $\gamma' = \gamma[\alpha \mapsto (R, \sigma_1, \sigma_2)]$. Then the result holds by inductive hypothesis since

$$((\emptyset, \emptyset) \mid \gamma') \in \mathcal{G}\, [\![!\Gamma, \alpha]\!]^j$$

□

LEMMA 37 (COMPAT TYPE APPLICATION).

$$\frac{!\Gamma \vdash e_1 \lesssim^{log} e_2 : \forall\alpha.\, \sigma'}{!\Gamma \vdash e_1\ [\sigma] \lesssim^{log} e_2\ [\sigma] : \sigma'[\sigma/\alpha]}$$

PROOF. Given $((\emptyset, \emptyset) \mid \gamma) \in \mathcal{G}\,[\![!\Gamma]\!]^j$, it is sufficient to show

$$((\gamma)_1(e_1\ [\sigma]), (\gamma)_2(e_2\ [\sigma])) \in \mathcal{E}\,[\![(\gamma)_R(\sigma'[\sigma/\alpha])]\!]^j$$

equivalently,

$$((\gamma)_1(e_1)\ [(\gamma)_1(\sigma)], (\gamma)_2(e_2)\ [(\gamma)_2(\sigma)]) \in \mathcal{E}\,[\![(\gamma)_R(\sigma')[(\gamma)_R(\sigma)/\alpha]]\!]^j$$

By Lemma 8 (Monadic Bind), and the definiton of $\mathcal{V}\,[\![\forall -.\, -]\!]^{j'}$, it is sufficient to prove for any $j' \leq j$ and

$$(\Lambda\alpha.\, v_1, \Lambda\alpha.\, v_2) \in \mathcal{V}\,[\![\forall\alpha.\, (\gamma)_R(\sigma')]\!]^{j'},$$

that

$$((\Lambda\alpha.\, v_1)\ [(\gamma)_1(\sigma)], (\Lambda\alpha.\, v_2)\ [(\gamma)_2(\sigma)]) \in \mathcal{E}\,[\![(\gamma)_R(\sigma')[(\gamma)_R(\sigma)/\alpha]]\!]^{j'}$$

If we define $\gamma' = \gamma[\alpha \mapsto (\mathcal{V}\,[\![(\gamma)_R(\sigma)]\!]^{-}, (\gamma)_1(\sigma), (\gamma)_2(\sigma))]$, then operational semantics dictates that

$$(\Lambda\alpha.\, v_1)\ [(\gamma)_1(\sigma)] \overset{U}{\hookrightarrow} (\gamma')_1(\sigma)$$
$$(\Lambda\alpha.\, v_2)\ [(\gamma)_2(\sigma)] \overset{U}{\hookrightarrow} (\gamma')_2(\sigma)$$

So it is sufficient to show

$$((\gamma')_1(v_1), (\gamma')_2(v_2)) \in \mathcal{V}\,[\![(\gamma)_R(\sigma')[(\gamma)_R(\sigma)/\alpha]]\!]^{j'-1}$$

By Lemma 3 (Monotonicity), we have $((\emptyset, \emptyset) \mid \gamma') \in \mathcal{G}\,[\![!\Gamma, \alpha]\!]^{j'-1}$, so by inductive hypothesis we have

$$((\gamma')_1(v_1), (\gamma')_2(v_2)) \in \mathcal{V}\,[\![(\gamma')_R(\sigma')]\!]^{j'-1}$$

So the result holds because we have

$$\mathcal{V}\,[\![(\gamma')_R(\sigma')]\!]^{j'-1} = \mathcal{V}\,[\![(\gamma)_R(\sigma')[(\gamma)_R(\sigma)/\alpha]]\!]^{j'-1}$$

by Lemma 40 (Compositionality). □

LEMMA 38 (FUNDAMENTAL LEMMA).

(1) *If* $!\Gamma \vdash_U v : \sigma$ *then* $!\Gamma \vdash_v v \lesssim^{log} v : \sigma$
(2) *If* $!\Gamma \vdash_U e : \sigma$ *then* $!\Gamma \vdash e \lesssim^{log} e : \sigma$
(3) *If* $\Psi; \Gamma \vdash_L s \mid e : \sigma$ *then* $\Gamma \vdash_L (s \mid e) \lesssim^{log} (s \mid e) : \sigma$

PROOF. By mutual induction on typing derivations, the cases are exactly the compatibility lemmas. □

### *2.5 Soundness*

LEMMA 39 (ADEQUACY).

(1) *If* $\cdot \vdash e_1 \lesssim^{log} e_2 : \sigma$, *then if* $e_1 \overset{U}{\hookrightarrow}{}^* v_1$ *there exists* $v_2$ *such that* $e_2 \overset{U}{\hookrightarrow}{}^* v_2$.

(2) *If* $\cdot \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma$*, then if* $(s_1 \mid e_1) \overset{L}{\hookrightarrow}^* (s_1' \mid v_1)$ *there exists* $(s_2' \mid v_2)$ *such that* $(s_2 \mid e_2) \overset{U}{\hookrightarrow}^* (s_2' \mid v_2)$.

Proof. Immediate from the definition. □

Lemma 40 (Compositionality).

(1) *If* $!\Gamma \vdash e_1 \lesssim^{log} e_2 : \sigma$*, then* $!\Gamma' \vdash C[e_1] \lesssim^{log} C[e_2] : \sigma'$.
(2) *If* $!\Gamma \vdash_v v_1 \lesssim^{log} v_2 : \sigma$*, then* $!\Gamma' \vdash C[v_1] \lesssim^{log} C[v_2] : \sigma'$.
(3) ...

Proof. By induction on contexts. The cases are exactly the compatibility lemmas. □

Theorem 1 (Soundness of Logical Relation). *In short,* $\lesssim^{log} \subset \lesssim^{ctx}$

Proof. Immediate corollary of Lemma 40 (Compositionality) and Lemma 39 (Adequacy). □

Lemma 41 (Eta Expansion for Functions).

$$\cdot \vdash_U v : \sigma_1 \to \sigma_2 \implies v \approx^{ctx}_{LU} \lambda(x : \sigma_1).\, v\ x$$

Proof. By Theorem 1 (Soundness of Logical Relation), sufficient to show for every $j$,

$$(v, \lambda(x : \sigma_1).\, v\ x) \in \mathcal{V}[\![\sigma_1 \to \sigma_2]\!]^j$$
$$(\lambda(x : \sigma_1).\, v\ x, v) \in \mathcal{V}[\![\sigma_1 \to \sigma_2]\!]^j$$

By Lemma 38 (Fundamental Lemma), $v = \lambda(x : \sigma_1).\, v\ x$ and given $j' \leq j$ and $(v_1, v_2) \in \mathcal{V}[\![\sigma_1]\!]^{j'}$, it is sufficient to show

$$(e[v_1/x], (\lambda(x : \sigma_1).\, v\ x)\ v_1) \in \mathcal{V}[\![\sigma_1 \to \sigma_2]\!]^j$$
$$((\lambda(x : \sigma_1).\, v\ x)\ v_1, e[v_1/x]) \in \mathcal{V}[\![\sigma_1 \to \sigma_2]\!]^j$$

But this after one reduction step we get related terms by Lemma 38 (Fundamental Lemma), so the result holds by Lemma 4 (Anti-Reduction). □

### 2.6 Copy/Share Cancellation

Lemma 42.

$$\frac{\mid\ !\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma}{\mid\ !\Gamma \vdash_L (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_1 : \Psi).\, e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\sigma]\!]^j}{((\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_1 : \Psi).\, v_1), (s_2 \mid v_2)) \in \mathcal{E}[\![\sigma]\!]^j}$$

$$\frac{\mid\ !\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : \sigma}{\mid\ !\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_2 : \Psi).\, e_2) : \sigma} \qquad \frac{((s_1 \mid v_1), (s_2 \mid v_2)) \in \mathcal{V}[\![\sigma]\!]^j}{((s_1 \mid v_1), (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_2 : \Psi).\, v_2)) \in \mathcal{E}[\![\sigma]\!]^j}$$

Proof. The open cases follow from the closed using Lemma 9 (Monadic Bind Under Share). For the closed cases, proceed by induction on $\sigma$. □

Corollary 1 (Copy-Share Cancellation).

$$!\Gamma \vdash_L (\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s : \Psi).\ e) \approx^{ctx} (s \mid e) : \sigma$$

Proof. From Lemma 38 (Fundamental Lemma), Theorem 1 (Soundness of Logical Relation), and Lemma 42. □

Lemma 43.

$$\frac{\mid\ !\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : !\sigma}{\mid\ !\Gamma \vdash_L (\emptyset \mid \mathsf{share}(s_1 : \Psi).\ \mathsf{copy}^{\sigma}\ e_1) \lesssim^{log} (s_2 \mid e_2) : !\sigma} \qquad \frac{((\emptyset \mid v_1), (\emptyset \mid v_2)) \in \mathcal{V}\,[\![!\sigma]\!]^j}{((\emptyset \mid \mathsf{share}\ \mathsf{copy}^{\sigma}\ v_1), (\emptyset \mid v_2)) \in \mathcal{E}\,[\![!\sigma]\!]^j}$$

$$\frac{\mid\ !\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (s_2 \mid e_2) : !\sigma}{\mid\ !\Gamma \vdash_L (s_1 \mid e_1) \lesssim^{log} (\emptyset \mid \mathsf{share}(s_2 : \Psi).\ \mathsf{copy}^{\sigma}\ e_2) : !\sigma} \qquad \frac{((\emptyset \mid v_1), (\emptyset \mid v_2)) \in \mathcal{V}\,[\![!\sigma]\!]^j}{((\emptyset \mid v_1), (\emptyset \mid \mathsf{share}\ \mathsf{copy}^{\sigma}\ v_2)) \in \mathcal{E}\,[\![!\sigma]\!]^j}$$

Proof. The open cases follow from the closed using Lemma 9 (Monadic Bind Under Share).

For the first closed case, assume

$$((\emptyset \mid v_1), (\emptyset \mid v_2)) \in \mathcal{V}\,[\![!\sigma]\!]^j$$

From the definition of $\mathcal{V}\,[\![\sigma]\!]^j$, this gives us that $s_1 = s_2 = \emptyset$ and

$$\exists \Psi_1, (s_1 \mid v_1'), \Psi_2, (s_2 \mid v_2').v_i = \mathsf{share}(s_i : \Psi_i).\ v_i' \wedge ((s_1 \mid v_1'), (s_2 \mid v_2')) \in \mathcal{V}\,[\![\sigma]\!]^j$$

We need to show

$$((\emptyset \mid \mathsf{share}\ \mathsf{copy}^{\sigma}\ \mathsf{share}(s_1 : \Psi_1).\ v_1'), (\emptyset \mid \mathsf{share}(s_2 : \Psi_2).\ v_2')) \in \mathcal{E}\,[\![!\sigma]\!]^j$$

By Lemma 42 (), $((\emptyset \mid \mathsf{copy}^{\sigma}\ \mathsf{share}(s_1 : \Psi_1).\ v_1'), (s_2 \mid v_2')) \in \mathcal{E}\,[\![\sigma]\!]^j$. Applying Lemma 9 (Monadic Bind Under Share), assume $j' \leq j$ and

$$((s_1' \mid v_1''), (s_2' \mid v_2'')) \in \mathcal{V}\,[\![\sigma]\!]^{j'} \tag{43}$$

Since $\mathcal{V}\,[\![\sigma]\!]^{j'}$ is defined over well-typed terms, there exist $\Psi_1'$ and $\Psi_2'$ such that $\Psi_i'; \cdot \vdash_L s_i' \mid v_i'' : (\sigma)_i$. It suffices to show

$$((\emptyset \mid \mathsf{share}(s_1' : \Psi_1).\ v_1''), (\emptyset \mid \mathsf{share}(s_2' : \Psi_2).\ v_2'')) \in \mathcal{E}\,[\![!\sigma]\!]^{j'}$$

Since $\mathcal{E}\,[\![!\sigma]\!]^{j'} \supseteq \mathcal{V}\,[\![!\sigma]\!]^{j'}$, this follows from the definition of $\mathcal{V}\,[\![!\sigma]\!]^{j'}$ and (43).

The second closed case is analagous. □

Corollary 2 (Share-Copy Cancellation).

$$!\Gamma \vdash_L (\emptyset \mid \mathsf{share}(s : \Psi).\ \mathsf{copy}^{\sigma}\ e) \approx^{ctx} (s \mid e) : !\sigma$$

Proof. From Lemma 38 (Fundamental Lemma), Theorem 1 (Soundness of Logical Relation), and Lemma 43. □